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Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes

Simone Gutt [a,*], John Rawnsley [b,1]

[a] *Département de Mathématiques, Université Libre de Bruxelles, Campus Plaine, CP 218, BE 1050 Brussels, Belgium*

[b] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK*

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Abstract

These notes grew out of the Quantisation Seminar 1997–1998 on Deligne's paper [P. Deligne, Déformations de l'algèbre des fonctions d'une variété symplectique: Comparaison entre Fedosov et De Wilde, Lecomte, Selecta Math. (New Series) 1 (1995) 667–697] and the lecture of the first author in the Workshop on Quantisation and Momentum Maps at the University of Warwick in December 1997.

We recall the definitions of the cohomology classes introduced by Deligne for equivalence classes of differential star products on a symplectic manifold and show the properties of and relations between these classes by elementary methods based on Čech cohomology. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we present in a purely Čech cohomology context some of the results given by Deligne [6] concerning cohomology classes associated to equivalence classes of differential star products on a symplectic manifold.

Star products were introduced in [1] to give a deformation approach to quantisation. A star product is a formal deformation of the algebraic structure of the space of smooth functions on

* Corresponding author. E-mail: sgutt@ulb.ac.be
[1] E-mail: j.rawnsley@warwick.ac.uk

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a Poisson manifold, both of the associative structure given by the usual product of functions and the Lie structure given by the Poisson bracket. We consider here only differential star products (i.e. defined by a series of bidifferential operators) on a symplectic manifold. Although the question makes sense more generally for Poisson manifolds, Deligne's method depends crucially on the Darboux theorem and the uniqueness of the Moyal star product on $\mathbb{R}^{2n}$ so the methods do not extend to general Poisson manifolds. Different methods are used by Kontsevich [17] to construct and classify differential star products on a Poisson manifold.

The existence of a differential star product on any symplectic manifold was first proven in 1983 by De Wilde and Lecomte [9] whilst the fact that equivalence classes of differential star products are parametrised by a series of elements in the second de Rham cohomology space of M appeared first in Nest and Tsygan [20], in Bertelson et al. [3,4] and in Deligne [6]. In the first two cited papers, the correspondence relies on the geometrical construction of a star product by Fedosov [10]; Fedosov takes a symplectic connection, extends it as a connection in the Weyl bundle whose curvature lies in the centre and builds from this a star product whose equivalence class is determined by the cohomology class of this central curvature. The classification then depends on showing that every differential star product is equivalent to a Fedosov star product.

In his paper, Deligne defines two cohomological classes associated to differential star products on a symplectic manifold. The first class is a relative class; fixing a star product on the manifold, it intrinsically associates to any equivalence class of star products an element in $H^2(M;\mathbb{R})[\![\nu]\!]$ (i.e. a series of elements in the second de Rham cohomology space of M). This is done in Čech cohomology by looking at the obstruction to gluing local equivalences (and is thus a globalisation of the old step by step techniques which showed that, at each order in the parameter, equivalence classes were parametrised by $H^2(M;\mathbb{R})$).

Deligne's second class is built from special local derivations of a star product. The same derivations played a special role in the first general existence theorem [9] for a star product on a symplectic manifold. Deligne used some properties of Fedosov's construction and central curvature class to relate his two classes and to see how to characterise an equivalence class of star products by the derivation related class and some extra data obtained from the second term in the deformation. We do this here by direct methods.

The content of our paper is as follows:

Section 2 includes definitions of star products and equivalence on symplectic manifolds as well as a brief study of the differential Hochschild cohomology of the algebra of smooth functions on a manifold.

Section 3 collects some basic results on the topological conditions for the equivalence of two star products. We determine when a self-equivalence is inner and what are the ν-linear derivations of a star product on a symplectic manifold (M,ω).

Section 4 describes the relative Čech cohomology class introduced by Deligne as the obstruction to piecing together local equivalences between two differential star products on a symplectic manifold.

Section 5 describes the intrinsic derivation-related Čech cohomology class associated to a star product; it is obtained by comparing local "ν-Euler" derivations of this star product. The relation between the relative class of two star products and their intrinsic derivation-related classes is found.

Section 6 introduces the characteristic class, defined from the intrinsic derivation-related class and the second term of the deformation. We show directly some equivariance properties of this class (relative to diffeomorphisms and to changes of the deformation parameter) and the fact that it characterises equivalence classes of star products. The proof of the fact that this class is the same as Fedosov's central curvature class is not included in these notes, see [6].

Section 7 includes the De Wilde proof [7] of the existence of a star product on any symplectic manifold. To whit a simultaneous construction of a star product and a family of local ν-Euler derivations of it yielding a given intrinsic derivation-related class. This employs the techniques of the previous sections, refining the Neroslavsky and Vlassov [19] step-by-step techniques to apply to the De Wilde–Lecomte proof [9].

Section 8 gives the first and second differential cohomology space for a star-deformed associative algebra viewed as an $\mathbb{R}[\![\nu]\!]$-algebra. In particular, it gives an elementary proof of the fact that the second differential Hochschild cohomology space for a star-deformed algebra $(C^\infty(M)[\![\nu]\!], *)$ is isomorphic to $Z^2(M;\mathbb{R}) \oplus \nu H^2(M;\mathbb{R})[\![\nu]\!]$, where $Z^2(M;\mathbb{R})$ is the space of closed 2-forms on the manifold (see also [25]).

Section 9 gives all automorphisms and derivations of a star product which are continuous for the ν-adic topology; in particular, we show that a symplectomorphism of a symplectic manifold (M,ω) can be extended to a ν-linear automorphism of a given differential star product on (M,ω) if and only if its action on $H^2(M;\mathbb{R})[\![\nu]\!]$ preserves its characteristic class.

Section 10 explains some of the steps to get from the Deligne's definition of a deformation [6] to the usual one considered in the first part of these notes. In his paper Deligne deduces this and other results from the algebraic geometrical approach to deformation theory; in these notes we give equivalent low-brow proofs based around partitions of unity and coverings by contractible Darboux charts to go between local and global structures.

Let us close the introduction by emphasising that the results in these pages are not new, except for Section 9, and can be found mostly in [6]. We decided to write these notes in view of the large number of people who asked for a written account of the seminar on the subject. The interest of the presentation is that it is self-contained and the proofs are done in an elementary way. Similar presentations of some of this material exist; in particular De Wilde [7] and Karabegov [16] give purely Čech-theoretic accounts of Deligne's intrinsic derivation-related class (see Section 5) and De Wilde shows by Čech methods how this class and a 2-form induced by the skew-symmetric part of the second term of the deformation characterise the equivalence class of the deformation.

2. Preliminaries

This section contains a basic introduction to the setting for the rest of the paper. It includes definitions of star products and equivalence on symplectic manifolds as well as a brief study of the differential Hochschild cohomology of the algebra of smooth functions on a manifold.

Definition 2.1. Let M be a smooth manifold then a *symplectic structure* on M is a closed 2-form ω on M which is non-degenerate as a bilinear form on each tangent space. A *symplectic manifold* is a pair (M, ω) consisting of a smooth manifold M together with a symplectic structure ω on M.

Definition 2.2. Let (M, ω) be a symplectic manifold then a *symplectic vector field* on M is a vector field X whose (local) flow preserves ω or, equivalently, if

$$\mathcal{L}_X \omega = 0.$$

The Cartan identity for the Lie derivative yields

$$\mathcal{L}_X \omega = i(X)\, \mathrm{d}\omega + \mathrm{d}(i(X)\omega) = \mathrm{d}(i(X)\omega)$$

since ω is closed. Hence X is symplectic if and only if $i(X)\omega$ is a closed 1-form.

Definition 2.3. If (M, ω) is a symplectic manifold then a vector field X for which $i(X)\omega$ is exact is called a *Hamiltonian vector field*. If $u \in C^\infty(M)$, then X_u denotes the unique Hamiltonian vector field with

$$i(X_u)\omega = \mathrm{d}u.$$

Obviously, the space of symplectic vector fields modulo the Hamiltonian vector fields is isomorphic to the space of closed 1-forms modulo the exact 1-forms and hence to $H^1(M; \mathbb{R})$. Locally the Poincaré lemma implies each symplectic vector field is a Hamiltonian vector field; as a consequence symplectic vector fields are also called *locally Hamiltonian vector fields*.

Definition 2.4. If (M, ω) is a symplectic manifold and $u, v \in C^\infty(M)$ then the *Poisson bracket* of u and v is defined by

$$\{u, v\} = X_u(v) = \omega(X_v, X_u).$$

The Poisson bracket makes $C^\infty(M)$ into a Lie algebra. The *Poisson tensor* Λ is the alternating 2-vector field with

$$\{u, v\} = \Lambda(\mathrm{d}u \wedge \mathrm{d}v).$$

Remark 2.5. In coordinates the components Λ^{ij} form the inverse matrix of the components ω_{ij} of ω. The Jacobi identity for the Poisson bracket Lie algebra is equivalent to the vanishing of the derivative $\mathrm{d}\omega$ or the Schouten bracket $[\Lambda, \Lambda]$.

In what follows we shall consider deformations of both the associative and Lie algebra structures of real-valued smooth functions $N = C^\infty(M)$; similar results hold for complex smooth functions. All deformations considered will be formal in the sense that they will be defined on $N[[\nu]]$ the space of formal power series in an indeterminate ν with coefficients in N. Questions of convergence of these formal series will not be considered.

Definition 2.6. [2]. A *star product* on (M, ω) is a bilinear map

$$N \times N \to N[\![\nu]\!], \qquad (u, v) \mapsto u * v = u *_\nu v = \sum_{r \geq 0} \nu^r C_r(u, v)$$

such that

- when the map is extended ν-linearly to $N[\![\nu]\!] \times N[\![\nu]\!]$ it is formally associative:

$$(u * v) * w = u * (v * w);$$

- (a) $C_0(u, v) = uv$, (b) $C_1(u, v) - C_1(v, u) = \{u, v\}$;
- $1 * u = u * 1 = u$.

Remark 2.7. In this definition we follow Deligne's normalisation for C_1, that its skew-symmetric part is $\frac{1}{2}\{\ ,\ \}$. In the original definition it was equal to the Poisson bracket.

Remark 2.8. Property (b) above implies that the centre of $N[\![\nu]\!]$, when the latter is viewed as an algebra with multiplication $*$, is a series whose terms Poisson commute with all functions so is an element of $\mathbb{R}[\![\nu]\!]$ when M is connected.

Definition 2.9. If $*$ is a star product on (M, ω) then we define the *star commutator* by

$$[u, v]_* = u * v - v * u,$$

which obviously makes $N[\![\nu]\!]$ into a Lie algebra with *star adjoint representation*

$$\mathrm{ad}_* u(v) = [u, v]_*.$$

Remark 2.10. Properties (a) and (b) of Definition 2.6 imply

$$[u, v]_* = \nu\{u, v\} + \cdots$$

so that repeated bracketing leads to higher and higher order terms. This makes $N[\![\nu]\!]$ an example of a *pronilpotent Lie algebra*, see Section 4 for some consequences of this.

Definition 2.11. Two star products $*$ and $*'$ on (M, ω) are said to be *equivalent* if there is a series

$$T = \mathrm{Id} + \sum_{r=1}^{\infty} \nu^r T_r,$$

where the T_r are linear operators on N, such that

$$T(f * g) = Tf *' Tg. \tag{1}$$

In studying star products on $N[\![\nu]\!]$ modulo equivalence we use the Gerstenhaber theory of deformations [12] of N which requires a knowledge of the Hochschild cohomology of N with values in N. So we begin by studying this.

A *p-cochain* on N is a p-linear map from $N \times \cdots \times N$ (p copies) to N. The Hochschild *coboundary operator* for the algebra N of smooth functions on a manifold M is denoted by ∂:

$$(\partial C)(u_0, \ldots, u_p) = u_0 C(u_1, \ldots, u_p) + \sum_{r=1}^{p} (-1)^r C(u_0, \ldots, u_{r-1}u_r, \ldots, u_p)$$
$$+ (-1)^{p-1} C(u_0, \ldots, u_{p-1}) u_p.$$

On 1- and 2-cochains ∂ is given by

$$(\partial F)(u, v) = uF(v) - F(uv) + F(u)v,$$
$$(\partial C)(u, v, w) = uC(v, w) - C(uv, w) + C(u, vw) - C(u, v)w.$$

A cochain C is called a *cocycle* if $\partial C = 0$, and a *coboundary* if $C = \partial B$ for some $(p-1)$-cochain B. A p-cochain C is called *differential* if it is given by differential operators on each argument and *k-differential* if the differential operators have order at most k. It is said to *vanish on constants* if it is zero whenever any argument is a constant function. 1-differential cochains vanishing on constants are always cocycles. 1-cocycles are derivations of $C^\infty(M)$, so are vector fields and hence are 1-differential cochains vanishing on constants. The Hochschild coboundary operator sends differential cochains to differential cochains.

Definition 2.12. The *pth differential Hochschild cohomology of N* is the space $H^p_{\mathrm{diff}}(N, N)$ of differential p-cocycles modulo differential p-coboundaries.

If C and D are p- and q-cochains, respectively, then we can define a $(p+q)$-cochain by

$$(C \otimes D)(u_1, \ldots, u_{p+q}) = C(u_1, \ldots, u_p) \cdot D(u_{p+1}, \ldots, u_{p+q}).$$

∂ acts as a graded derivation

$$\partial(C \otimes D) = \partial C \otimes D + (-1)^p C \otimes \partial D.$$

If D is a differential operator of order k, then we may view it as a k-differential 1-cochain. For a vector field X we have $\partial X = 0$, and if $k \geq 2$ a repeated application of Leibniz' rule then shows that ∂D is a bi-differential operator of order $k - 1$.

We define the support $\operatorname{supp} C$ of a cochain C to be the union of the supports of its coefficients when written in coordinates.

Proposition 2.13. *If C is a 1-differential p-cochain on $\mathbb{R}^n$ and A is its alternating part then $C = \partial B + A$ where B is 2-differential, and determined by C so that* $\operatorname{supp} B \subset \operatorname{supp} C$.

Proof. If C is a 1-differential p-cochain vanishing on constants, then C has the form

$$C(u_1, \ldots, u_p) = \sum_{i_1, \ldots, i_p} C_{i_1, \ldots, i_p} \frac{\partial u_1}{\partial x_{i_1}} \cdots \frac{\partial u_p}{\partial x_{i_p}}$$

where the coefficients are given by

$$C_{i_1,\dots,i_p} = C(x_{i_1},\dots,x_{i_p}).$$

If σ is a permutation of $\{1,\dots,p\}$ and C a p-cochain then we set

$$(\sigma\cdot C)(u_1,\dots,u_p) = C(u_{\sigma^{-1}(1)},\dots,u_{\sigma^{-1}(p)})$$

which is an action on cochains. It is not, however, compatible with the Hochschild coboundary.

If τ is a transposition of consecutive integers, say the interchange of i and $i+1$ and C a 1-differential p-cochain vanishing on constants then we define a $(p-1)$-cochain $\Phi_\tau(C)$ by

$$\Phi_\tau(C)(u_1,\dots,u_{p-1}) = (-1)^i \sum_{r,s} C(u_1,\dots,u_{i-1},x_r,x_s,u_{i+1},\dots,u_{p-1})\frac{\partial^2 u_i}{\partial x_r\partial x_s}.$$

Then, using the Leibniz formula for second derivatives and the derivation property of C in each argument, a straightforward computation shows that

$$\partial\Phi_\tau(C) = C + \tau\cdot C.$$

If τ_1 and τ_2 are each transpositions of consecutive integers then we have

$$\partial(\Phi_{\tau_1}(\tau_2\cdot C) - \Phi_{\tau_2}(C)) = C - \tau_1\tau_2\cdot C.$$

It is clear that if we keep composing such transpositions we build up any element σ of the symmetric group and we manufacture a 2-differential $(p-1)$-cochain $\Phi_\sigma(C)$ which is completely determined by C once we fix a decomposition of σ into a product of transpositions of consecutive integers and

$$\partial\Phi_\sigma(C) = C - \mathrm{sign}(\sigma)\sigma\cdot C.$$

If we set

$$\Phi(C) = \frac{1}{p!}\sum_{\sigma\in S_p}\Phi_\sigma(C)$$

then

$$C = \partial\Phi(C) + \frac{1}{p!}\sum_{\sigma\in S_p}\mathrm{sign}(\sigma)\sigma\cdot C$$

so that C is cohomologous to its skewsymmetric part.

Note that the explicit nature of Φ means that $\operatorname{supp}\Phi(C)\subset\operatorname{supp}C$. □

Proposition 2.14. *If C is a differential p-cocycle on $C^\infty(\mathbb{R}^n)$ then there is a differential $(p-1)$-cochain B and a skewsymmetric 1-differential p-cocycle A with $C = \partial B + A$. If*

C vanishes on constants then B (and hence A) can be chosen to vanish on constants. We can choose B and A so that supp B *and* supp A *are contained in* supp C.

Proof. Any 1-cocycle is a vector field so the result is trivially true for $p = 1$.

Assume the result true for r-cocycles with $r < p$ and let C be a differential p-cocycle with $p \geq 2$. Consider $C(u_1, \ldots, u_p)$ as a differential operator in u_1. Suppose it has order $k > 1$ then we shall show that we can subtract a coboundary to reduce the order. An induction then shows the order can be reduced to 1.

We consider the terms of highest order in u_1

$$C(u_1, \ldots, u_p) = \sum_{i_1, \ldots, i_k} \frac{\partial^k u_1}{\partial x_{i_1} \cdots \partial x_{i_k}} D_{i_1, \ldots, i_k}(u_2, \ldots, u_p) + \cdots$$

where the $D_{i_1, \ldots, i_k}$ are $(p-1)$-cochains, symmetric in $i_1, \ldots, i_k$. In other words, in multi-index notation $i = (i_1, \ldots, i_k)$,

$$C = \sum_{|i|=k} \partial_i \otimes D_i + \cdots$$

It follows from the derivation property with respect to tensor products above that

$$\partial C = -\sum_{|i|=k} \partial_i \otimes \partial D_i + \cdots$$

so that, when C is a p-cocycle, the coefficients of the highest order derivatives of u_1 are $(p-1)$-cocycles. By induction $D_i = \partial E_i + F_i$ with F_i a skewsymmetric 1-differential cocycle and supports in supp C if needed. Set

$$G = \sum_{|i|=k} \partial_i \otimes E_i$$

then an easy calculation gives

$$C' \stackrel{\text{def}}{=} C + \partial G = \sum_{|i|=k} \partial_i \otimes F_i + H$$

where H only has terms involving derivatives of the first argument of order strictly less than k and C' is still a cocycle.

Taking the coboundary of this equation we have

$$0 = \sum_{|i|=k} \partial(\partial_i) \otimes F_i + \partial H,$$

and since $\partial(\partial_i)(u, v)$, for $|i| = k$, only has terms $\partial_{i'} u \partial_{i''} v$ with $|i'| + |i''| = k$ and both $|i'|$, $|i''|$ non-zero, the highest order terms in the first argument which can occur are of order $k-1$ and these terms in $\partial(\partial_i) \otimes F_i$ will be of order 1 in the second and remaining arguments. So the leading terms in $\partial(\partial_i) \otimes F_i$ are of multi-order $(k-1, 1, \ldots, 1)$. We examine how such terms can arise from ∂H.

If we expand $H = \sum_{i_1,\dots,i_{p+1}} H_{i_1\dots i_{p+1}} \partial_{i_1} \otimes \cdots \otimes \partial_{i_{p+1}}$, where each i_r is a multiindex and the coefficients are symmetric in each multiindex separately, then

$$\partial H = \sum_{i_1,\dots,i_{p+1}} H_{i_1\dots i_{p+1}} \sum_r (-1)^{r-1} \partial_{i_1} \otimes \cdots \otimes \partial(\partial_{i_r}) \cdots \partial_{i_{p+1}}$$

and so terms of order $(k-1,1,\dots,1)$ can only come from labels $\{i_1,\dots,i_r,\dots,i_{p+1}\}$ where $|i_1| = k-1$, exactly one multiindex i_r has $|i_r| = 2$ for some $r \geq 2$ and all other multiindices have length 1.

Thus, if we take the terms of order $(k-1,1,\dots,1)$ in the equation and write out all multiindices fully, we have

$$kF_{(i_1\cdots i_{k-1} j_1),j_2,\dots,j_p} + 2H_{(i_1\cdots i_{k-1}),(j_1 j_2),\dots,j_p} + \cdots + 2H_{(i_1\cdots i_{k-1}),j_1,\dots,(j_{p-1} j_p)} = 0$$

with the H terms symmetric in the bracketed pairs of indices. Denoting by S_p the group of permutations of $(1,\dots,p)$ and $\epsilon(\sigma)$ the signature of a permutation σ, if we antisymmetrise over the last p indices, all the H terms drop out and we get a relation among the F's

$$\sum_{\sigma\in S_p} \epsilon(\sigma) F_{(i_1\cdots i_{k-1} j_{\sigma(1)}),j_{\sigma(2)},\dots,j_{\sigma(p)}} = 0$$

which implies, since F is skewsymmetric in its last $p-1$ indices,

$$F_{(i_1\cdots i_{k-1} j_1),j_2,\dots,j_p} = \sum_{s=2}^{p} (-1)^s F_{(i_1\cdots i_{k-1} j_s),j_1,\dots,\widehat{j_s},\dots,j_p}$$

where $\widehat{\cdot}$ denotes omission and, since F is symmetric in its first k labels,

$$kF_{(i_1\cdots i_k),j_2,\dots,j_p} = \sum_{r=1}^{k}\sum_{s=2}^{p} (-1)^s F_{(i_1\cdots\widehat{i_r}\cdots i_k j_s),i_r,j_2,\dots,\widehat{j_s},\dots,j_p}.$$

Thus

$$(k+p-1)F_{(i_1\cdots i_k),j_2,\dots,j_p} = \sum_{s=2}^{p} (-1)^s K_{i_1\cdots i_k,j_s,j_2,\dots,\widehat{j_s},\dots,j_p}$$

where

$$K_{i_1\cdots i_k,j_s,j_2,\dots,\widehat{j_s},\dots,j_p} = \sum_{r=1}^{k} F_{(i_1\cdots\widehat{i_r}\cdots i_k j_s),i_r,j_2,\dots,\widehat{j_s},\dots,j_p} + F_{(i_1\cdots i_k),j_s,j_2,\dots,\widehat{j_s},\dots,j_p}$$

is symmetric in its first $k+1$ indices.

One can write

$$\begin{aligned}(k+p-1)&F_{(i_1\cdots i_k),j_2,\dots,j_p}\\ &= \sum_{s=2}^{p} (-1)^s K_{i_1\cdots i_k,j_s,j_2,\dots,\widehat{j_s},\dots,j_p}\end{aligned}$$

$$= \sum_{s=2}^{p} (-1)^s \left[(-1)^{s-2} K_{i_1 \cdots i_k, j_2, j_3, \dots, j_p} + \sum_{t=3}^{s} (-1)^{s-t} (K_{i_1 \cdots i_k, j_t, j_2, \dots, j_{t-1}, \widehat{j_t}, \dots, j_p} \right.$$

$$\left. + K_{i_1 \cdots i_k, j_{t-1}, j_2, \dots, \widehat{j_{t-1}}, j_t, \dots, j_p}) \right]$$

$$= (p-1) K_{i_1 \cdots i_k, j_2, j_3, \dots, j_p}$$

$$+ \sum_{t=3}^{p} (-1)^t (p+1-t) (K_{i_1 \cdots i_k, j_t, j_2, \dots, j_{t-1}, \widehat{j_t}, \dots, j_p} + K_{i_1 \cdots i_k, j_{t-1}, j_2, \dots, \widehat{j_{t-1}}, j_t, \dots, j_p})$$

so that the terms corresponding to the first line in the last equation coincide with the terms of order k in the first variable of the coboundary of a constant multiple of

$$K_{i_1 \cdots i_k, j_2, j_3, \dots, j_p} \partial_{i_1 \cdots i_k j_2} \otimes \partial_{j_3} \otimes \cdots \otimes \partial_{j_p}$$

and the terms corresponding to each summand in the second line in the last equation coincide with the terms of order k in the first variable of the coboundary of a constant multiple of

$$(K_{i_1 \cdots i_k, j_t, j_2, \dots, j_{t-1}, \widehat{j_t}, \dots, j_p} + K_{i_1 \cdots i_k, j_{t-1}, j_2, \dots, \widehat{j_{t-1}}, j_t, \dots, j_p})$$
$$\times \partial_{i_1 \cdots i_k} \otimes \partial_{j_2} \otimes \cdots \otimes \partial_{j_{t-1} j_t} \otimes \cdots \otimes \partial_{j_p}.$$

Combining the above results, we can build a $p-1$ cochain G' so that $C - \partial(G')$ is a multi differential operator with terms involving derivatives of the first argument of order less than k. Iterating, we can reduce the order in the first argument to 1.

Now assume that

$$C = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes D_i$$

then

$$\partial C = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes \partial D_i$$

so C being a cocycle is equivalent to the D_i being cocycles. In this case we have $D_i = \partial E_i + F_i$ with F_i 1-differential and

$$C + \partial \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes E_i = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes F_i$$

and the RHS is now 1-differential in all arguments.

Using the previous Proposition, this 1-differential cocycle is equal to its total skewsymmetrization plus a coboundary.

Hence the induction proceeds. □

Theorem 2.15. *Every differential p-cocycle C on a manifold M is the sum of the coboundary of a differential ($p-1$)-cochain and a 1-differential skew-symmetric p-cocycle A:*

$$C = \partial B + A.$$

If C vanishes on constants then B can be chosen to vanish on constants also.

Proof. Take a locally finite covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of M by charts with a subordinate partition of unity ρ_λ. Then any p-cocycle C is a locally finite sum of p-cocycles

$$C = \sum_{\lambda\in\Lambda} \rho_\lambda C.$$

with supports in charts. By Propositions 2.13 and 2.14

$$\rho_\lambda C = \partial B_\lambda + A_\lambda,$$

with the supports of B_λ and A_λ in U_λ. It follows that the sums

$$B = \sum_{\lambda\in\Lambda} B_\lambda, \qquad A = \sum_{\lambda\in\Lambda} A_\lambda$$

are locally finite, so globally defined, and $C = \partial B + A$ as required. □

Corollary 2.16. $H^p_{\mathrm{diff}}(N, N) = \Gamma(\bigwedge^p TM)$.

Proof. It remains to show that the alternating part of a coboundary is zero and we leave this to the reader. □

Remark 2.17. This is a smooth version of the Hochschild–Kostant–Rosenberg theorem [15]. It was first mentioned in the context of smooth functions by Vey [24] who considered the proof well-known. Other classes of cochains than differential have been considered, such as distributional cochains, with essentially the same result [22]. But for completely general cochains the full cohomology is not known.

Remark 2.18. The proofs of the above results work globally on a manifold if we use a connection to write the cochain in terms of its symbol and do the induction with respect to the degree. Then we see that all the choices can be made explicit, and the inductions are finite, so the method can be made constructive.

Remark 2.19. In the symplectic case the 1-differential skew-symmetric p-cocycle A in Theorem 2.15 can be rewritten in terms of the Hamiltonian vector fields and a smooth p-form α as

$$C(u_1, \ldots, u_p) = (\partial B)(u_1, \ldots, u_p) + \alpha(X_{u_1}, \ldots, X_{u_p}). \tag{2}$$

Definition 2.20. A star product $*$ on (M, ω) is called *differential* if the 2-cochains $C_r(u, v)$ giving it are bidifferential operators.

Definition 2.21. Two differential star products $*$ and $*'$ on (M, ω) are said to be *differentially equivalent* if there is a series

$$T = \mathrm{Id} + \sum_{r=1}^{\infty} \nu^r T_r,$$

where the T_r are differential operators on N, such that

$$T(f * g) = Tf *' Tg. \tag{3}$$

In fact for differential star products there is no difference between the two notions of equivalence as the following result shows [6,18]:

Theorem 2.22. *Let $*$ and $*'$ be differential star products and $T(u) = u + \sum_{r\geq 1} \nu^r T_r(u)$ an equivalence so that $T(u * v) = T(u) *' T(v)$ then the T_r are differential operators.*

Proof. Suppose we know that the first k operators $T_1, \ldots, T_k$ in T are differential operators and set $T'(u) = u + \sum_{1\leq r\leq k} \nu^r T_r(u)$. Then $T'' = T'^{-1} \circ T$ is an equivalence between the differential star products $*$ and $*''$, where $u *'' v = T'^{-1}(T'(u) *' T'(v))$. T'' has the form $T''(u) = u + \nu^{k+1} T''_{k+1}(u) + \cdots$ Taking the terms of degree $k+1$ in $u * v = T''^{-1}(T''(u) *'' T''(v))$ we see that $(\partial T''_{k+1})(u, v) = T''_{k+1}(u)v + uT''_{k+1}(v) - T''_{k+1}(uv)$ is a bidifferential symmetric 2-cocycle. By Theorem 2.15 $\partial T''_{k+1}$ is the coboundary of a differential 1-cochain plus a 1-differential skew-symmetric cochain. Since both exact terms are symmetric, the skew-symmetric term vanishes. Thus there is a differential 1-cochain B such that $\partial(T''_{k+1} - B) = 0$. It follows that $X = T''_{k+1} - B$ is a derivation of N and hence is a vector field. Thus $T''_{k+1} = B + X$ is differential. T_{k+1} is a combination of $T_1, \ldots, T_k$ and T''_{k+1} and hence is also differential. It follows now by induction that T is differential. □

A simple application of Theorem 2.15 is as follows:

Proposition 2.23. *A differential star product is equivalent to one with linear term in ν given by $\frac{1}{2}\{u, v\}$.*

Proof. Let $u * v = uv + \nu C_1(u, v) + \cdots$ be a star product then $C_1(u, v)$ is a Hochschild cocycle with antisymmetric part given by $\frac{1}{2}\{u, v\}$. By Theorem 2.15 $C_1(u, v) = \frac{1}{2}\{u, v\} + uB(v) - B(uv) + B(u)v$ for a differential 1-cochain B. If we set $T(u) = u + \nu B(u)$ and $u *' v = T(T^{-1}(u) * T^{-1}(v))$ then an easy calculation gives $u *' v = uv + \frac{1}{2}\nu\{u, v\} + \cdots$ T is obviously a differential equivalence so that $*'$ is differential. □

3. Local equivalences and ν-linear derivations

In this section we collect some basic results on the topological conditions for the equivalence of two star products [18]; when is a self-equivalence inner; and what are the ν-linear derivations of a star product on a symplectic manifold (M, ω).

Proposition 3.1. *Let $*$ and $*'$ be two differential star products on (M,ω) and suppose that $H^2(M;\mathbb{R})=0$. Then there exists a local equivalence $T=\mathrm{Id}+\sum_{k\geq 1}\nu^k T_k$ on $N[\![\nu]\!]$ such that $u*'v=T(T^{-1}u*T^{-1}v)$ for all $u,v\in N[\![\nu]\!]$.*

Proof. Let us suppose that, modulo some equivalence, the two star products $*$ and $*'$ coincide up to order k. Then associativity at order k shows that $C_k-C'_k$ is a Hochschild 2-cocycle and so by (2) can be written as

$$(C_k-C'_k)(u,v)=(\partial B)(u,v)+A(X_u,X_v)$$

for a 2-form A. The total skew-symmetrisation of the associativity relation at order $k+1$ shows that A is a closed 2-form. Since the second cohomology vanishes, A is exact, $A=\mathrm{d}F$. Transforming by the equivalence defined by $Tu=u+\nu^{k-1}2F(X_u)$, we can assume that the skew-symmetric part of $C_k-C'_k$ vanishes. Then $(C_k-C'_k)(u,v)=(\partial B)(u,v)=uB(v)-B(uv)+B(u)v$, where B is a differential operator on N and using the equivalence defined by $T=I+\nu^k B$ we can assume that the star products coincide, modulo an equivalence, up to order $k+1$.

This gives the inductive step, and since two star products always agree in their leading term, it follows, by induction, that they are equivalent. □

Corollary 3.2. *Let $*$ and $*'$ be two differential star products on (M,ω). Let U be a contractible open subset of M and $N_U=C^\infty(U)$. Then there exists a local equivalence $T=\mathrm{Id}+\sum_{k\geq 1}\nu^k T_k$ on $N_U[\![\nu]\!]$ that $u*'v=T(T^{-1}u*T^{-1}v)$ for all $u,v\in N_U[\![\nu]\!]$.*

Proof. A contractible open set has vanishing cohomology groups and a differentiable star product on M restricts to give a star product on any open set U of M, so Proposition 3.1 can be applied. □

Proposition 3.3. *Let $*$ be a differential star product on (M,ω) and suppose that $H^1(M;\mathbb{R})$ vanishes. Then any self-equivalence $A=\mathrm{Id}+\sum_{k\geq 1}\nu^k A_k$ of $*$ is inner: $A=\exp\,\mathrm{ad}_* a$ for some $a\in N[\![\nu]\!]$.*

Proof. We build a recursively. The condition $A(u*v)=Au*Av$ implies (taking the coefficient of ν) that $A_1(uv)+C_1(u,v)=A_1(u)v+uA_1(v)+C_1(u,v)$ so that A_1 is a vector field. Taking the skew part of the terms in ν^2 we have that A_1 is a derivation of the Poisson bracket. It follows that $A_1(u)=\{a_0,u\}$ for some function a_0. Then $(\exp-\mathrm{ad}_* a_0)\circ A=\mathrm{Id}+O(\nu^2)$ as an easy calculation shows. Now we proceed by induction.

Suppose we have found $a^{(k-1)}=a_0+\cdots+\nu^{k-1}a_{k-1}$ such that

$$A'=(\exp-\mathrm{ad}_* a^{(k-1)})\circ A=\mathrm{Id}+\nu^{k+1}A'_{k+1}+O(\nu^{k+2})$$

then we can repeat the argument of Proposition 3.3 since A' also preserves $*$. The terms of degree $k+1$ show that A'_{k+1} is a vector field and the antisymmetric part of the terms of degree $k+2$ show that it is a Hamiltonian vector field $A'_{k+1}(u)=\{a_k,u\}$ for some function a_k. Taking $a^{(k)} = a^{(k-1)} + \nu^k a_k$ gives a formal function with $(\exp - \mathrm{ad}_* a^{(k)}) \circ A = \mathrm{Id} + \mathrm{O}(\nu^{k+2})$ completing the induction step. □

Again this yields directly:

Corollary 3.4. *Let $*$ be a differential star product on (M, ω). Let U be a contractible open set of M and $N_U = C^\infty(U)$. If $A = \mathrm{Id} + \sum_{k\geq 1} \nu^k A_k$ is a formal linear operator on $N_U[\![\nu]\!]$ which preserves the differential star product $*$, then there is $a \in N_U[\![\nu]\!]$ with $A = \exp \mathrm{ad}_* a$.*

Proposition 3.5. *Any ν-linear derivation of a differential star product $*$ on (M, ω) is of the form $D = \sum_{i\geq 0} \nu^i D_i$ where D_i corresponds to a symplectic vectorfield X_i and is given on a contractible open set U by*

$$D_i u|_U = \frac{1}{\nu}(f_i^U * u - u * f_i^U)$$

if $X_i u|_U = \{f_i^U, u\}|_U$.

Proof. Assuming a derivation D is of the form $Du = \nu^k D'u + \cdots$ (where $k \geq 0$), the equation $D(u * v) = Du * v + u * Dv$ at order k in ν yields $D'(u.v) = D'u.v + u.D'v$, so $D'u = Xu$, where X is a vector field on M. Taking the skew part of terms in ν^{k+1}, we have that X is a symplectic vector field on M, i.e. that $\mathcal{L}_X \omega = 0$. In that case, one can write, on a contractible open set U,

$$Xu|_U = \{f^U, u\}|_U, \quad \text{where } f^U \in C^\infty(U).$$

Since the C_r vanish on constants for $r > 0$, one can globally define $E : N \to N[\![\nu]\!]$ by

$$E(u) = \frac{1}{\nu}(f^U * u - u * f^U)$$

and by associativity of $*_\nu$, E is a derivation of the star product. Notice that $D - \nu^k E$ starts with the terms of order $k + 1$. An induction gives the result. □

Corollary 3.6. *With the same notation as in Corollary 3.4, any local ν-linear derivation D_U of a differential star product $*$ on $N_U[\![\nu]\!]$ for a contractible open set U is essentially inner: $D_U = (1/\nu)\,\mathrm{ad}_* d_U$ for some $d_U \in N_U[\![\nu]\!]$.*

4. The relative class

We shall describe here the Čech cohomology class introduced by Deligne when one considers two differential star products on a symplectic manifold. It is built from local equivalences between those star products, using the property that any local self-equivalence of a differential star product is of the form $\exp \mathrm{ad}_* a$ for some locally defined a.

For convenience, we write the composition of automorphisms of the form $\exp \mathrm{ad}_* a$ in terms of a. In a pronilpotent situation this is done with the *Campbell–Baker–Hausdorff composition* which is denoted by $a \circ_* b$:

$$a \circ_* b = a + \int_0^1 \psi(\exp \mathrm{ad}_* a \circ \exp t\, \mathrm{ad}_* b) b \, \mathrm{d}t,$$

where

$$\psi(z) = \frac{z \log(z)}{z - 1} = \sum_{n \geq 1} \left(\frac{(-1)^n}{n+1} + \frac{(-1)^{n-1}}{n} \right) (z-1)^n.$$

Notice that the formula is well defined (at any given order in ν, only a finite number of terms arise) and it is given by the usual series

$$a \circ_* b = a + b + \tfrac{1}{2}[a, b]_* + \tfrac{1}{12}([a, [a, b]_*]_* + [b, [b, a]_*]_*) \cdots$$

The following results are standard [5].

Lemma 4.1.
- $\circ_*$ *is an associative composition law;*
- $\exp \mathrm{ad}_*(a \circ_* b) = \exp \mathrm{ad}_* a \circ \exp \mathrm{ad}_* b$;
- $a \circ_* b \circ_* (-a) = \exp(\mathrm{ad}_* a) b$;
- $-(a \circ_* b) = (-b) \circ_* (-a)$;
- $\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_0 (-a) \circ_* (a + tb) = \dfrac{1 - \exp(-\mathrm{ad}_* a)}{\mathrm{ad}_* a}(b).$

Let (M, ω) be a symplectic manifold. We fix a locally finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ by Darboux coordinate charts such that the U_α and all their non-empty intersections are contractible, and we fix a partition of unity $\{\theta_\alpha\}_{\alpha \in I}$ subordinate to $\mathcal{U}$. Set $N_\alpha = C^\infty(U_\alpha)$, $N_{\alpha\beta} = C^\infty(U_\alpha \cap U_\beta)$, and so on.

Now suppose that $*$ and $*'$ are two differential star products on (M, ω). We have seen that their restrictions to $N_\alpha[\![\nu]\!]$ are equivalent so there exist formal differential operators $T_\alpha : N_\alpha[\![\nu]\!] \to N_\alpha[\![\nu]\!]$ such that

$$T_\alpha(u * v) = T_\alpha(u) *' T_\alpha(v), \quad u, v \in N_\alpha[\![\nu]\!].$$

On $U_\alpha \cap U_\beta$, $T_\beta^{-1} \circ T_\alpha$ will be a self-equivalence of $*$ on $N_{\alpha\beta}[\![\nu]\!]$ and so there will be elements $t_{\beta\alpha} = -t_{\alpha\beta}$ in $N_{\alpha\beta}[\![\nu]\!]$ with

$$T_\beta^{-1} \circ T_\alpha = \exp \mathrm{ad}_* t_{\beta\alpha}.$$

On $U_\alpha \cap U_\beta \cap U_\gamma$ the element

$$t_{\gamma\beta\alpha} = t_{\alpha\gamma} \circ_* t_{\gamma\beta} \circ_* t_{\beta\alpha}$$

induces the identity automorphism and hence is in the centre $\mathbb{R}[\![\nu]\!]$ of $N_{\alpha\beta\gamma}[\![\nu]\!]$. The family of $t_{\gamma\beta\alpha}$ is thus a Čech 2-cocycle for the covering $\mathcal{U}$ with values in $\mathbb{R}[\![\nu]\!]$. The standard arguments show that its class does not depend on the choices made, and is compatible with refinements. Since every open cover has a refinement of the kind considered it follows that $t_{\gamma\beta\alpha}$ determines a unique Čech cohomology class $[t_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\nu]\!]$.

Definition 4.2.

$$t(*',*) = [t_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\nu]\!]$$

is *Deligne's relative class*.

Proposition 4.3. *If $*$, $*'$, $*''$ are three differential star products on (M,ω) then*

$$t(*'',*) = t(*'',*') + t(*',*). \tag{4}$$

Proof. Let the local equivalences between $*'$ and $*$ be T_α and between $*''$ and $*'$ be S_α. On $U_\alpha \cap U_\beta$ let

$$T_\beta^{-1} \circ T_\alpha = \exp \operatorname{ad}_* t_{\beta\alpha}, \qquad S_\beta^{-1} \circ S_\alpha = \exp \operatorname{ad}_{*'} s_{\beta\alpha}$$

and set $V_\alpha = S_\alpha \circ T_\alpha$ which are local equivalences between $*''$ and $*$. Then

$$V_\beta^{-1} \circ V_\alpha = \exp \operatorname{ad}_*(t_{\beta\alpha} \circ_* T_\alpha^{-1}(s_{\beta\alpha}))$$

and hence we can choose

$$v_{\beta\alpha} = t_{\beta\alpha} \circ_* T_\alpha^{-1}(s_{\beta\alpha}).$$

Observe that $v_{\beta\alpha} = -v_{\alpha\beta}$ since

$$\begin{aligned} t_{\beta\alpha} \circ_* T_\alpha^{-1}(s_{\beta\alpha}) \circ_* (-t_{\beta\alpha}) &= \exp \operatorname{ad}_* t_{\beta\alpha}(T_\alpha^{-1}(s_{\beta\alpha})) \\ &= T_\beta^{-1} T_\alpha T_\alpha^{-1}(s_{\beta\alpha}) = -T_\beta^{-1}(s_{\alpha\beta}); \end{aligned}$$

then, on $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$\begin{aligned} T_\gamma^{-1}(s_{\alpha\gamma}) \circ_* t_{\gamma\beta} &= t_{\gamma\beta} \circ_* (-t_{\gamma\beta}) \circ_* T_\gamma^{-1}(s_{\alpha\gamma}) \circ_* t_{\gamma\beta} \\ &= t_{\gamma\beta} \circ_* \exp \operatorname{ad}_* -t_{\gamma\beta}(T_\gamma^{-1}(s_{\alpha\gamma})) \\ &= t_{\gamma\beta} \circ_* (T_\beta^{-1} \circ T_\gamma \circ T_\gamma^{-1})(s_{\alpha\gamma}) \\ &= t_{\gamma\beta} \circ_* T_\beta^{-1}(s_{\alpha\gamma}) \end{aligned}$$

so

$$\begin{aligned} v_{\gamma\beta\alpha} &= v_{\alpha\gamma} \circ_* v_{\gamma\beta} \circ_* v_{\beta\alpha} \\ &= t_{\alpha\gamma} \circ_* T_\gamma^{-1}(s_{\alpha\gamma}) \circ_* t_{\gamma\beta} \circ_* T_\beta^{-1}(s_{\gamma\beta}) \circ_* t_{\beta\alpha} \circ_* T_\alpha^{-1}(s_{\beta\alpha}) \\ &= t_{\alpha\gamma} \circ_* t_{\gamma\beta} \circ_* t_{\beta\alpha} \circ_* T_\alpha^{-1}(s_{\alpha\gamma} \circ_* s_{\gamma\beta} \circ_* s_{\beta\alpha}) \\ &= t_{\gamma\beta\alpha} \circ_* T_\alpha^{-1}(s_{\gamma\beta\alpha}) \\ &= t_{\gamma\beta\alpha} \circ_* s_{\gamma\beta\alpha} \\ &= t_{\gamma\beta\alpha} + s_{\gamma\beta\alpha}, \end{aligned}$$

the last two steps following since $s_{\gamma\beta\alpha}$ is central. □

Proposition 4.4. *The class $t(*', *)$ vanishes if and only if the two differential star products are equivalent.*

Proof. If $*$ and $*'$ are equivalent, the equivalence being defined by T, we can choose the restriction of T to U_α for T_α and the class obviously vanishes.

Now consider what happens if the class $t(*', *)$ vanishes. Then we can modify $t_{\beta\alpha}$ by addition of a central element (and so not changing its adjoint action) so that $t_{\gamma\beta\alpha} = 0$. But then $t_{\gamma\beta}$ is a cocycle and hence a coboundary, so there are functions $t_\alpha \in N_\alpha[\![\nu]\!]$ with $t_{\gamma\beta} = (-t_\gamma) \circ_* t_\beta$.

This is shown by the following standard inductive argument. At order zero, the cocycle condition is

$$t^0_{\alpha\gamma} + t^0_{\gamma\beta} + t^0_{\beta\alpha} = 0.$$

Defining

$$t^{\{0\}}_\beta = \sum_\alpha \theta_\alpha t^0_{\alpha\beta},$$

yields

$$t_{\gamma\beta} = (-t^{\{0\}}_\gamma) \circ_* t^{\{0\}}_\beta$$

up to order one in ν. If there is a solution

$$t^{\{k\}}_\alpha = \sum_{r \le k} \nu^r t^r_\alpha$$

so that

$$t_{\gamma\beta} = (-t^{\{k\}}_\gamma) \circ_* t^{\{k\}}_\beta$$

up to order $k+1$, then the cocycle

$$c_{\gamma\beta} = -t^{\{k\}}_\gamma \circ_* t_{\gamma\beta} \circ_* t^{\{k\}}_\beta$$

has first non-vanishing term of order $k+1$ with

$$c^{k+1}_{\alpha\gamma} + c^{k+1}_{\gamma\beta} + c^{k+1}_{\beta\alpha} = 0.$$

Defining

$$t^{k+1}_\beta = \sum_\alpha \theta_\alpha c^{k+1}_{\alpha\beta}$$

and

$$t^{\{k+1\}}_\alpha = t^{\{k\}}_\alpha + \nu^{k+1} t^{k+1}_\alpha.$$

we have

$$t_{\gamma\beta} = (-t_{\gamma}^{\{k+1\}}) \circ_* t_{\beta}^{\{k+1\}}$$

up to order $k+2$ in ν, hence the result by induction. Setting

$$T'_{\alpha} = T_{\alpha} \circ \exp \mathrm{ad}_* t_{\alpha},$$

we have $T'_{\alpha} = T'_{\beta}$ on $U_{\alpha} \cap U_{\beta}$ and hence there is a global equivalence T' on N with $T' = T'_{\alpha}$ on U_{α}. □

Given a differential star product $*$ we thus have a map from equivalence classes of differential star products to $H^2(M; \mathbb{R})[\![\nu]\!]$ given by

$$[*'] \mapsto t(*', *)$$

and we have just shown that this map is injective.

Proposition 4.5. *Fixing a differential star product $*$, the map from equivalence classes of differential star products to $H^2(M; \mathbb{R})[\![\nu]\!]$ given by*

$$[*'] \mapsto t(*', *)$$

is surjective.

Proof. To see that the map is surjective we proceed in two steps. We first show that given a 2-cocycle $t_{\gamma\beta\alpha}$, we can find $t_{\beta\alpha} \in N_{\alpha\beta}[\![\nu]\!]$ so that

$$t_{\gamma\beta\alpha} = t_{\alpha\gamma} \circ_* t_{\gamma\beta} \circ_* t_{\beta\alpha}.$$

We then construct a differential operator T_{α} on U_{α} starting with the identity so that $T_{\beta}^{-1} \circ T_{\alpha} = \exp \mathrm{ad}_* t_{\beta\alpha}$.

For the first step, we use, as above, the fact that the sheaf of functions is fine. At order zero, the cocycle condition takes the form

$$t^0_{\gamma\beta\alpha} - t^0_{\delta\beta\alpha} + t^0_{\delta\gamma\alpha} - t^0_{\delta\gamma\beta} = 0.$$

We define

$$t^0_{\beta\alpha} = \sum_{\gamma} \theta_{\gamma} t^0_{\gamma\beta\alpha}$$

so that

$$t^0_{\gamma\beta\alpha} = t^0_{\alpha\gamma} + t^0_{\gamma\beta} + t^0_{\beta\alpha}.$$

Assume now, by induction, that we have $t^{\{k\}}_{\beta\alpha}$ such that $t^{\{k\}}_{\beta\alpha} = -t^{\{k\}}_{\alpha\beta}$ and

$$t_{\gamma\beta\alpha} = t^{\{k\}}_{\alpha\gamma} \circ_* t^{\{k\}}_{\gamma\beta} \circ_* t^{\{k\}}_{\beta\alpha}$$

up to order $k+1$. Let us define

$$c_{\gamma\beta\alpha} = t_{\alpha\gamma}^{\{k\}} \circ_* t_{\gamma\beta}^{\{k\}} \circ_* t_{\beta\alpha}^{\{k\}}$$

so that the coefficients $c^r_{\gamma\beta\alpha}$ are constants $\forall r \le k$. Note that in a non-commutative situation c is not necessarily a cocycle. Nevertheless the $c^{k+1}_{\gamma\beta\alpha}$ are completely skew-symmetric in their indices since

$$c^{k+1}_{\alpha\gamma\beta} = (t_{\beta\alpha}^{\{k\}} \circ_* c_{\gamma\beta\alpha} \circ_* t_{\alpha\beta}^{\{k\}})^{k-1} = c^{k-1}_{\gamma\beta\alpha}$$

and

$$\begin{aligned} c_{\gamma\alpha\beta} &= t_{\beta\gamma}^{\{k\}} \circ_* t_{\gamma\alpha}^{\{k\}} \circ_* t_{\alpha\beta}^{\{k\}} \\ &= (-t_{\gamma\beta}^{\{k\}}) \circ_* (-t_{\alpha\gamma}^{\{k\}}) \circ_* (-t_{\beta\alpha}^{\{k\}}) \\ &= -(t_{\beta\alpha}^{\{k\}} \circ_* t_{\alpha\gamma}^{\{k\}} \circ_* t_{\gamma\beta}^{\{k\}}) = -c_{\alpha\gamma\beta}. \end{aligned}$$

Furthermore, it can be checked that

$$(c_{\gamma\beta\alpha} \circ c_{\delta\beta\alpha} \circ c_{\delta\gamma\alpha} \circ c_{\delta\gamma\beta})^{k-1} = c^{k+1}_{\gamma\beta\alpha} - c^{k-1}_{\delta\beta\alpha} + c^{k+1}_{\delta\gamma\alpha} - c^{k-1}_{\delta\gamma\beta} = 0.$$

Hence, if we define

$$t^{k+1}_{\beta\alpha} = \sum_\gamma \theta_\gamma (t^{k+1}_{\gamma\beta\alpha} - c^{k-1}_{\gamma\beta\alpha})$$

and

$$t_{\beta\alpha}^{\{k+1\}} = t_{\beta\alpha}^{\{k\}} + \nu^{k+1} t^{k-1}_{\beta\alpha},$$

then we have

$$t_{\gamma\beta\alpha} = t_{\alpha\gamma}^{\{k+1\}} \circ_* t_{\gamma\beta}^{\{k+1\}} \circ_* t_{\beta\alpha}^{\{k+1\}}$$

up to order $k+2$, and $t_{\beta\alpha}^{\{k+1\}} = -t_{\alpha\beta}^{\{k-1\}}$ so the induction proceeds.

For the second step, we need to find differential operators T_α on $N_\alpha[\![\nu]\!]$, starting with the identity, so that $\exp \operatorname{ad}_* t_{\beta\alpha} = T_\beta^{-1} \circ T_\alpha$. This, again, is a recursive argument: Suppose we have found $T_\alpha^{(k)}$ such that

$$T_\beta^{(k)} \circ \exp \operatorname{ad}_* t_{\beta\alpha} \circ T_\alpha^{(k)^{-1}} = \operatorname{Id} + \nu^k S_{\beta\alpha} + \cdots,$$

then it is easy to see that $S_{\beta\alpha}$ is a 1-cocycle with values in the smooth differential operators vanishing on constants. Since these form a fine sheaf, this 1-cocycle is a coboundary, so there are differential operators S_α on $N_\alpha[\![\nu]\!]$ vanishing on constants with $S_{\beta\alpha} = S_\alpha - S_\beta$ on $U_\alpha \cap U_\beta$. Setting

$$T_\alpha^{(k+1)} = (\operatorname{Id} + \nu^k S_\alpha) \circ T_\alpha^{(k)},$$

we have

$$T_\beta^{(k+1)} \circ \exp \operatorname{ad}_* t_{\beta\alpha} \circ T_\alpha^{(k+1)^{-1}} = \operatorname{Id} + \nu^{k+1} S'_{\beta\alpha} + \cdots$$

and the recursion proceeds. Having found such operators T_α we then twist $*$ to yield $*'$ by defining

$$u *' v = T_\alpha(T_\alpha^{-1}(u) * T_\alpha^{-1}(v)) \quad \text{on } U_\alpha.$$

From the way we constructed $*'$ it is easy to see that the class of $*'$ will be $t(*', *)$. □

We summarise these results in a theorem.

Theorem 4.6. *Fixing a differential star product $*$, the relative class $t(*', *)$ of any other differential star product $*'$ in $H^2(M; \mathbb{R})[\![\nu]\!]$ depends only on the equivalence class of $*'$, and this sets up a bijection between the set of equivalence classes of differential star products and $H^2(M; \mathbb{R})[\![\nu]\!]$.*

5. The intrinsic derivation-related class

The addition formula of Proposition 4.3 suggests that $t(*', *)$ should be a difference of classes $c(*'), c(*) \in H^2(M; \mathbb{R})[\![\nu]\!]$. Moreover by Proposition 4.5 the class $c(*)$ should determine the star product $*$ up to equivalence. As a step in that direction we consider an intrinsic class which is an obstruction to piecing together local derivations of the star product.

We retain the notation of Section 4 and continue to denote by $\mathcal{U}$ the covering by contractible Darboux charts.

Definition 5.1. Say that a derivation D of $N[\![\nu]\!], *$ is *ν-Euler* on an open set U if it has the form

$$D = \nu\frac{\partial}{\partial\nu} + X + D', \tag{5}$$

where X is conformally symplectic ($\mathcal{L}_X\omega = \omega$) and $D' = \sum_{r\geq 1} \nu^r D'_r$ with the D'_r differential operators on U.

Proposition 5.2. *Let $*$ be a differential star product on (M, ω) then for each $U_\alpha \in \mathcal{U}$ we have a ν-Euler derivation $D_\alpha = \nu(\partial/\partial\nu) + X_\alpha + D'_\alpha$ of the algebra $(N_\alpha[\![\nu]\!], *)$.*

Proof. On an open set in $\mathbb{R}^{2n}$ with the standard symplectic structure Ω denote the Poisson bracket by P. Let X be a conformal vector field so $\mathcal{L}_X\Omega = \Omega$ and hence that the Poisson tensor P satisfies $\mathcal{L}_X P = -P$. It follows that the power P^r of P as a bidifferential operator satisfies $\mathcal{L}_X P^r = -rP^r$. The Moyal star product $*^{\mathrm{M}}$ is given by $u *^{\mathrm{M}} v = uv + \sum_{r\geq 1}(\nu/2)^r/r!P^r(u, v)$. It is easy to see that $D = \nu(\partial/\partial\nu) + X$ is a derivation of $*^{\mathrm{M}}$.

(U_α, ω) is symplectomorphic to an open set in $\mathbb{R}^{2n}$ and any differential star product on this open set is equivalent to $*^{\mathrm{M}}$. We can then pull back D and $*^{\mathrm{M}}$ to U_α by a symplectomorphism to give a star product $*'$ with a derivation of the form $\nu(\partial/\partial\nu) + X_\alpha$. If T is an equivalence of $*$ with $*'$ on U_α then $D_\alpha = T^{-1} \circ (\nu(\partial/\partial\nu) + X_\alpha) \circ T$ is a derivation of the required form. $\square$

We take such a collection of derivations D_α given by Proposition 5.2 and on $U_\alpha \cap U_\beta$ we consider the differences $D_\beta - D_\alpha$. They are derivations of $*$ and the ν derivatives cancel out, so $D_\beta - D_\alpha$ is a ν-linear derivation of $N_{\alpha\beta}[\![\nu]\!]$. Any ν-linear derivation is of the form $(1/\nu)\,\mathrm{ad}_* d$, so there are $d_{\beta\alpha} \in N_{\alpha\beta}[\![\nu]\!]$ with

$$D_\beta - D_\alpha = \frac{1}{\nu}\,\mathrm{ad}_* d_{\beta\alpha}, \tag{6}$$

with $d_{\beta\alpha}$ unique up to a central element. On $U_\alpha \cap U_\beta \cap U_\gamma$ the combination $d_{\alpha\gamma} + d_{\gamma\beta} + d_{\beta\alpha}$ must be central and hence defines $d_{\gamma\beta\alpha} \in \mathbb{R}[\![\nu]\!]$. It is easy to see that $d_{\gamma\beta\alpha}$ is a 2-cocycle whose Čech class $[d_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\nu]\!]$ does not depend on any of the choices made.

Definition 5.3. $d(*) = [d_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\nu]\!]$ is *Deligne's intrinsic derivation-related class*.

Remark 5.4. In fact the class considered by Deligne is actually $(1/\nu)d(*)$ but we prefer the present normalisation. De Wilde [7] and Karabegov [16] give purely Čech-theoretic accounts of this class.

Proposition 5.5. *If $*$ and $*'$ are equivalent differential star products then $d(*') = d(*)$.*

Proof. Suppose $T(u * v) = T(u) *' T(v)$, and we have chosen local ν-Euler derivations D_α for $*$ then we can take $D'_\alpha = T D_\alpha T^{-1}$ for $*'$. Then

$$D'_\beta - D'_\alpha = T(D_\beta - D_\alpha)T^{-1} = T\left(\frac{1}{\nu}\,\mathrm{ad}_* d_{\beta\alpha}\right)T^{-1} = \frac{1}{\nu}\,\mathrm{ad}_{*'} T(d_{\beta\alpha}).$$

Thus, for these choices of derivations, $d'_{\beta\alpha} = T(d_{\beta\alpha})$ and so $d'_{\gamma\beta\alpha} = T(d_{\gamma\beta\alpha}) = d_{\gamma\beta\alpha}$ since the higher order terms of T are differential operators vanishing on constants. $\square$

Proposition 5.6. *If $d(*) = \sum_{r\geq 0} \nu^r d^r(*)$ then $d^0(*) = [\omega]$ under the de Rham isomorphism, and $d^1(*) = 0$.*

Proof. For d^0, consider the terms of degree zero in (6) using (5) applied to a function $u \in N_\alpha$

$$(X_\beta - X_\alpha)u = \{d^0_{\beta\alpha}, u\}.$$

We set $\theta_\alpha = i(X_\alpha)\omega$ then $\mathrm{d}\theta_\alpha = \omega$ on U_α. Hence

$$\begin{aligned}(\theta_\beta - \theta_\alpha)(X_u) &= \omega(X_\beta - X_\alpha, X_u)\\ &= -(X_\beta - X_\alpha)u = -\{d^0_{\beta\alpha}, u\} = X_u(d^0_{\beta\alpha})\end{aligned}$$

so $\theta_\beta - \theta_\alpha = \mathrm{d}(d^0_{\beta\alpha})$ on $U_\alpha \cap U_\beta$. Hence $d^0_{\gamma\beta\alpha}$ is the Čech representative corresponding with the closed 2-form ω under the de Rham isomorphism. Thus $d^0(*) = [\omega]$.

For d^1 we first observe that by Proposition 5.5 we can replace $*$ by any equivalent star product. In particular, we may assume that $*$ has $C_1(u, v) = \frac{1}{2}\{u, v\}$ and the antisymmetric part C_2^- of C_2 is given by a closed 2-form A, $C_2^-(u, v) = A(X_u, X_v)$. Now

$$\begin{aligned}\operatorname{ad}_* d_{\beta\alpha} u &= d_{\beta\alpha} * u - u * d_{\beta\alpha}\\ &= \nu\{d_{\beta\alpha}, u\} + 2\nu^2 C_2^-(d_{\beta\alpha}, u) + \cdots\\ &= \nu\{d^0_{\beta\alpha}, u\} + \nu^2(\{d^1_{\beta\alpha}, u\} + 2C_2^-(d^0_{\beta\alpha}, u)) + O(\nu^3)\end{aligned}$$

for $u \in N_{\alpha\beta}$. Equating terms of degree one in (6) we have

$$\begin{aligned}(D^{(1)}_\beta - D^{(1)}_\alpha)u &= \{d^1_{\beta\alpha}, u\} + 2C_2^-(d^0_{\beta\alpha}, u)\\ &= \{d^1_{\beta\alpha}, u\} + 2A(X_{d^0_{\beta\alpha}}, X_u).\end{aligned} \tag{7}$$

If we take the terms of degree one in ν in $D_\alpha(u * v) = D_\alpha(u) * v + u * D_\alpha(v)$ we see at once that $D^{(1)}_\alpha = Y_\alpha$ is a vector field. If we take the antisymmetric part (in u and v) of the terms of degree two in ν we obtain

$$\begin{aligned}&2C_2^-(u, v) + X_\alpha(C_2^-(u, v)) - C_2^-(X_\alpha(u), v) - C_2^-(u, X_\alpha(v))\\ &\quad = \tfrac{1}{2}\{D^{(1)}_\alpha(u), v\} + \tfrac{1}{2}\{u, D^{(1)}_\alpha(v)\} - \tfrac{1}{2}D^{(1)}_\alpha(\{u, v\}),\end{aligned}$$

which can be rewritten in terms of the closed 2-form A as

$$2(\mathcal{L}_{X_\alpha} A)(X_u, X_v) = \{Y_\alpha(u), v\} + \{u, Y_\alpha(v)\} - Y_\alpha(\{u, v\}). \tag{8}$$

If we let $B_\alpha = -i(Y_\alpha)\omega$ then $B_\alpha(X_u) = Y_\alpha(u)$ so the right-hand side of (8) becomes

$$-X_v(B_\alpha(X_u)) + X_u(B_\alpha(X_v)) - B_\alpha(X_{\{u,v\}}) = \mathrm{d}B_\alpha(X_u, X_v)$$

so

$$\mathcal{L}_{X_\alpha} A = \tfrac{1}{2}\,\mathrm{d}B_\alpha.$$

Since A is closed, then from Cartan's identity, $i(X_\alpha)A - \frac{1}{2}B_\alpha$ is closed on U_α and hence exact. There is thus a smooth function f_α on U_α with

$$i(X_\alpha)A - \tfrac{1}{2}B_\alpha = \mathrm{d}f_\alpha.$$

Substituting into (7), and using $X_{d^0_{\beta\alpha}} = X_\beta - X_\alpha$ we have

$$\begin{aligned}(Y_\beta - Y_\alpha)u &= \{d^1_{\beta\alpha}, u\} + 2A(X_\beta - X_\alpha, X_u)\\ &= \{d^1_{\beta\alpha}, u\} + B_\beta(X_u) - B_\alpha(X_u) + 2\,\mathrm{d}f_\beta(X_u) - 2\,\mathrm{d}f_\alpha(X_u).\end{aligned}$$

Since $Y_\beta(u) = B_\beta(X_u)$, these terms cancel leaving

$$\{d^1_{\beta\alpha}, u\} = 2\{f_\beta - f_\alpha, u\}$$

and hence the class $d^1(*) = 0$. $\square$

Lemma 5.7. *Consider two differentiable star products $*$ and $*'$ on (M,ω) with local equivalences T_α and local ν-Euler derivations D_α for $*$. Then $D'_\alpha = T_\alpha \circ D_\alpha \circ T_\alpha^{-1}$ are local ν-Euler derivations for $*'$. Let $D_\beta - D_\alpha = (1/\nu)\,\mathrm{ad}_*\, d_{\beta\alpha}$ and $T_\beta^{-1} \circ T_\alpha = \exp \mathrm{ad}_*\, t_{\beta\alpha}$ on $U_\alpha \cap U_\beta$. Then $D'_\beta - D'_\alpha = (1/\nu)\mathrm{ad}_{*'} d'_{\beta\alpha}$ where*

$$d'_{\beta\alpha} = T_\beta d_{\beta\alpha} - \nu T_\beta \circ \left(\frac{1-\exp(-\,\mathrm{ad}_*\, t_{\alpha\beta})}{\mathrm{ad}_*\, t_{\alpha\beta}}\right) \circ D_\alpha t_{\alpha\beta}.$$

Proof. Since, for any (local) derivation D and any (locally defined) $t \in N[\![\nu]\!]$, one has $D \circ \mathrm{ad}_*\, t = \mathrm{ad}_*(Dt) + \mathrm{ad}_*\, t \circ D$ so that inductively

$$D \circ \mathrm{ad}^n_*\, t = \mathrm{ad}^n_*\, t \circ D + \sum_{i=0}^{n-1} \mathrm{ad}^i_*\, t \circ \mathrm{ad}_*(Dt) \circ \mathrm{ad}^{n-1-i}_*\, t$$

and

$$\begin{aligned} \mathrm{e}^{-\mathrm{ad}_* t} \circ D \circ \mathrm{e}^{\mathrm{ad}_* t} &= D + \mathrm{e}^{-\mathrm{ad}_* t} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_0 \mathrm{e}^{\mathrm{ad}_*(t - sDt)} \\ &= D + \left(\frac{1-\mathrm{e}^{-\mathrm{ad}(\mathrm{ad}_* t)}}{\mathrm{ad}(\mathrm{ad}_* t)}\right) \mathrm{ad}_*\, Dt \\ &= D + \mathrm{ad}_*\left(\left(\frac{1-\exp(-\,\mathrm{ad}_*\, t)}{\mathrm{ad}_*\, t}\right) Dt\right). \end{aligned}$$

One gets

$$\begin{aligned} D'_\beta - D'_\alpha &= T_\beta \circ D_\beta \circ T_\beta^{-1} - T_\alpha \circ D_\alpha \circ T_\alpha^{-1} \\ &= T_\beta \circ (D_\beta - (T_\beta^{-1} T_\alpha) \circ D_\alpha \circ (T_\alpha^{-1} T_\beta)) \circ T_\beta^{-1} \\ &= T_\beta \circ (D_\beta - D_\alpha) \circ T_\beta^{-1} \\ &\quad - T_\beta \circ \left(\mathrm{ad}_*\left[\left(\frac{1-\exp(-\,\mathrm{ad}_*\, t_{\alpha\beta})}{\mathrm{ad}_*\, t_{\alpha\beta}}\right) D_\alpha t_{\alpha\beta}\right]\right) \circ T_\beta^{-1} \\ &= T_\beta \circ \mathrm{ad}_*\left(\frac{1}{\nu} d_{\beta\alpha} - \left(\frac{1-\exp(-\,\mathrm{ad}_*\, t_{\alpha\beta})}{\mathrm{ad}_*\, t_{\alpha\beta}}\right) D_\alpha t_{\alpha\beta}\right) \circ T_\beta^{-1} \\ &= \mathrm{ad}_{*'} \frac{1}{\nu} d'_{\beta\alpha}. \end{aligned}$$

for the above defined $d'_{\beta\alpha}$.

Notice that $d'_{\alpha\beta} = -d'_{\beta\alpha}$. Indeed,

$$\begin{aligned} d'_{\alpha\beta} &= T_\alpha d_{\alpha\beta} - \nu T_\alpha \circ \left(\frac{1-\exp(-\,\mathrm{ad}_*\, t_{\beta\alpha})}{\mathrm{ad}_*\, t_{\beta\alpha}}\right) \circ D_\beta t_{\beta\alpha} \\ &= T_\alpha d_{\alpha\beta} - \nu T_\beta (T_\beta^{-1} T_\alpha) \circ \left(\frac{1-\exp(-\,\mathrm{ad}_*\, t_{\beta\alpha})}{\mathrm{ad}_*\, t_{\beta\alpha}}\right) \circ D_\beta t_{\beta\alpha} \\ &= T_\alpha d_{\alpha\beta} - \nu T_\beta \left(\frac{\exp(\mathrm{ad}_*\, t_{\beta\alpha}) - 1}{\mathrm{ad}_*\, t_{\beta\alpha}}\right) D_\beta t_{\beta\alpha} \end{aligned}$$

$$
\begin{aligned}
&= T_\alpha d_{\alpha\beta} + \nu T_\beta \left(\frac{1 - \exp(-\operatorname{ad}_* t_{\alpha\beta})}{\operatorname{ad}_* t_{\alpha\beta}} \right) D_\alpha t_{\alpha\beta} \\
&\quad + \nu T_\beta \left(\frac{1 - \exp(-\operatorname{ad}_* t_{\alpha\beta})}{\operatorname{ad}_* t_{\alpha\beta}} \right) (D_\beta - D_\alpha) t_{\alpha\beta}
\end{aligned}
$$

Since $\nu(D_\beta - D_\alpha)t_{\alpha\beta} = \operatorname{ad}_* d_{\beta\alpha} t_{\alpha\beta} = -\operatorname{ad}_* t_{\alpha\beta} d_{\beta\alpha}$, the above gives

$$
\begin{aligned}
\mathrm{d}'_{\alpha\beta} &= T_\alpha d_{\alpha\beta} + \nu T_\beta \left(\frac{1 - \exp(-\operatorname{ad}_* t_{\alpha\beta})}{\operatorname{ad}_* t_{\alpha\beta}} \right) D_\alpha t_{\alpha\beta} \\
&\quad - T_\beta d_{\beta\alpha} + T_\beta \exp(\operatorname{ad}_* t_{\beta\alpha}) d_{\beta\alpha} \\
&= -d'_{\beta\alpha} + T_\alpha d_{\alpha\beta} + T_\beta T_\beta^{-1} T_\alpha d_{\beta\alpha} = -d'_{\beta\alpha}. \qquad \square
\end{aligned}
$$

Lemma 5.8. *In the situation of the lemma above and with the same notation*

$$
\mathrm{d}'_{\gamma\beta\alpha} = T_\alpha \left(d_{\gamma\beta\alpha} + \nu^2 \frac{\partial}{\partial \nu} t_{\gamma\beta\alpha} \right).
$$

Proof. We shall use the formula for the derivative of the exponential map:

$$
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (-a) \circ_* (a + tb) = \frac{1 - \exp(-\operatorname{ad}_* a)}{\operatorname{ad}_* a}(b).
$$

This also yields

$$
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (c \circ_* -a) \circ_* (a + tb) \circ_* (-c) = \exp \operatorname{ad}_* c \frac{1 - \exp(-\operatorname{ad}_* a)}{\operatorname{ad}_* a}(b).
$$

Remark that if D is any (locally defined) derivation, one has

$$
D(a \circ_* b) = \left.\frac{\mathrm{d}}{\mathrm{d}s}(a + sDa) \circ_* b\right|_0 + \left.\frac{\mathrm{d}}{\mathrm{d}s}(a) \circ_* (b + sDb)\right|_0 .
$$

It follows that

$$
\begin{aligned}
\nu \frac{\partial}{\partial \nu} t_{\gamma\beta\alpha} &= D_\gamma (t_{\alpha\gamma} \circ_* t_{\gamma\beta} \circ_* t_{\beta\alpha}) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \big((t_{\alpha\gamma} + sD_\gamma t_{\alpha\gamma}) \circ_* t_{\gamma\beta} \circ_* t_{\beta\alpha}\big) \\
&\quad + \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \big(t_{\alpha\gamma} \circ_* (t_{\gamma\beta} + sD_\gamma t_{\gamma\beta}) \circ_* t_{\beta\alpha}\big) \\
&\quad + \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \big(t_{\alpha\gamma} \circ_* t_{\gamma\beta} \circ_* (t_{\beta\alpha} + sD_\gamma t_{\beta\alpha})\big) \\
&= \exp \operatorname{ad}_* t_{\alpha\gamma} \frac{1 - \exp(-\operatorname{ad}_* t_{\alpha\gamma})}{\operatorname{ad}_* t_{\alpha\gamma}} D_\gamma t_{\alpha\gamma} \\
&\quad + \exp \operatorname{ad}_* t_{\alpha\beta} \frac{1 - \exp(-\operatorname{ad}_* t_{\gamma\beta})}{\operatorname{ad}_* t_{\gamma\beta}} D_\gamma t_{\gamma\beta} \\
&\quad + \frac{1 - \exp(-\operatorname{ad}_* t_{\beta\alpha})}{\operatorname{ad}_* t_{\beta\alpha}} D_\gamma t_{\beta\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1-\exp(-\operatorname{ad}_* t_{\gamma\alpha})}{\operatorname{ad}_* t_{\gamma\alpha}}\right) D_\gamma t_{\alpha\gamma} \\
&\quad + T_\alpha^{-1} T_\beta \left(\frac{1-\exp(-\operatorname{ad}_* t_{\gamma\beta})}{\operatorname{ad}_* t_{\gamma\beta}}\right) D_\gamma t_{\gamma\beta} \\
&\quad + \left(\frac{1-\exp(-\operatorname{ad}_* t_{\beta\alpha})}{\operatorname{ad}_* t_{\beta\alpha}}\right) D_\beta t_{\beta\alpha} \\
&\quad + \left(\frac{1-\exp(-\operatorname{ad}_* t_{\beta\alpha})}{\operatorname{ad}_* t_{\beta\alpha}}\right) (D_\gamma - D_\beta) t_{\beta\alpha}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
d'_{\alpha\gamma} - d'_{\beta\gamma} - d'_{\alpha\beta} &= -T_\alpha d_{\alpha\beta} + \nu T_\alpha \left(\frac{1-\exp(-\operatorname{ad}_* t_{\beta\alpha})}{\operatorname{ad}_* t_{\beta\alpha}}\right) D_\beta t_{\beta\alpha} \\
&\quad - T_\beta d_{\beta\gamma} + \nu T_\beta \left(\frac{1-\exp(-\operatorname{ad}_* t_{\gamma\beta})}{\operatorname{ad}_* t_{\gamma\beta}}\right) D_\gamma t_{\gamma\beta} \\
&\quad + T_\alpha d_{\alpha\gamma} - \nu T_\alpha \left(\frac{1-\exp(-\operatorname{ad}_* t_{\gamma\alpha})}{\operatorname{ad}_* t_{\gamma\alpha}}\right) D_\gamma t_{\gamma\alpha}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
d'_{\gamma\alpha} - d'_{\beta\gamma} - d'_{\alpha\beta} &= T_\alpha (\nu^2 \frac{\partial}{\partial\nu} t_{\gamma\beta\alpha}) - T_\alpha d_{\alpha\beta} - T_\beta d_{\beta\gamma} \\
&\quad + T_\alpha d_{\alpha\gamma} + T_\alpha (1 - T_\alpha^{-1} T_\beta) d_{\gamma\beta} \\
&= T_\alpha (\nu^2 \frac{\partial}{\partial\nu} t_{\gamma\beta\alpha} + d_{\alpha\gamma} + d_{\beta\alpha} + d_{\gamma\beta}).
\end{aligned}
$$

This proves the theorem: □

Theorem 5.9. *The relative and intrinsic derivation-related classes of two differentiable star products $*$ and $*'$ are related by*

$$
\nu^2 \frac{\partial}{\partial\nu} t(*', *) = d(*') - d(*). \tag{9}
$$

6. The characteristic class

Formula (9) of Theorem 5.9 gives the relation between the relative and intrinsic derivation-related classes of two differential star products $*$ and $*'$. It shows that the information which is "lost" in $d(*') - d(*)$ corresponds to the zeroth order term in ν of $t(*', *)$. We compute below what is this missing part.

Take two differential star products

$$
\begin{aligned}
u * v &= u.v + \nu C_1(u, v) + \nu^2 C_2(u, v) + \cdots, \\
u *' v &= u.v + \nu C'_1(u, v) + \nu^2 C'_2(u, v) + \cdots
\end{aligned}
$$

Since, by associativity at order 1, $C_1' - C_1$ is a symmetric Hochschild 2-cocycle, we write

$$C_1'(u, v) = C_1(u, v) - u.E_1(v) - E_1(u).v + E_1(u.v),$$

where E_1 is a differential operator on N defined up to a vector field. Then, by associativity at order 2 and 3 of the two star products

$$C'^{-}_2(u, v) = C^{-}_2(u, v) - \tfrac{1}{2}[\{u, E_1(v)\} + \{E_1(u), v\} - E_1(\{u, v\})] + A(X_u, X_v),$$

where A is a closed 2-form on M. Notice that the de Rham class $[A]$ of A does not depend on the choice of E_1. We write

$$[A] = (C'^{-}_2 - C^{-}_2)^{\#}.$$

A local equivalence T_α on U_α so that

$$u *' v|_{U_\alpha} = T_\alpha(T_\alpha^{-1}u * T_\alpha^{-1}v)$$

is given by $T_\alpha = \mathrm{Id} + \nu E_\alpha + \cdots$, where

$$E_\alpha(u) = E_1(u) + B_\alpha(X_u),$$

with $A = -\frac{1}{2}\,\mathrm{d}B_\alpha$ on U_α. The Čech class corresponding to $[A]$ is calculated from 1-forms F_α on U_α with $A = \mathrm{d}F_\alpha$ on U_α. If $F_\beta - F_\alpha = \mathrm{d}f_{\beta\alpha}$ on $U_\alpha \cap U_\beta$ then $a_{\gamma\beta\alpha} = f_{\alpha\gamma} + f_{\gamma\beta} + f_{\beta\alpha}$ is constant on $U_\alpha \cap U_\beta \cap U_\gamma$ and $[A] = [a_{\gamma\beta\alpha}]$. Here we can take $F_\alpha = -1/2B_\alpha$. So

$$\begin{aligned}\exp \mathrm{ad}_* t_{\beta\alpha} &= T_\beta^{-1} \circ T_\alpha \\ &= \mathrm{Id} + \nu(E_\alpha - E_\beta) + \cdots \\ &= \mathrm{Id} + \nu(B_\alpha - B_\beta) + \cdots\end{aligned}$$

hence

$$\begin{aligned}\{t^0_{\beta\alpha}, u\} &= 2(F_\beta - F_\alpha)(X_u) \\ &= 2\,\mathrm{d}f_{\beta\alpha}(X_u) \\ &= -2\{f_{\beta\alpha}, u\}\end{aligned}$$

and finally

$$[t^0_{\gamma\beta\alpha}] = -2[A].$$

Remark 6.1. In [14,8] it was shown that any bidifferential operator C, vanishing on constants, which is a 2-cocycle for the Chevalley cohomology of $(N, \{\ ,\ \})$ with values in N associated to the adjoint representation (i.e. such that

$$S_{u,v,w}[\{u, C(v, w)\} - C(\{u, v\}, w)] = 0,$$

where $S_{u,v,w}$ denotes the sum over cyclic permutations of u, v and w) can be written as

$$C(u, v) = aS^3_\Gamma(u, v) + A(X_u, X_v) + [\{u, Ev\} + \{Eu, v\} - E(\{u, v\})],$$

where $a \in \mathbb{R}$, where S^3_Γ is a bidifferential 2-cocycle vanishing on constants which is never a coboundary and whose symbol is of order 3 in each argument, where A is a closed 2-form on M and where E is a differential operator vanishing on constants. Hence

$$H^2_{\mathrm{Chev,nc}}(N, N) = \mathbb{R} \oplus H^2(M; \mathbb{R})$$

and the # operator is the projection on the second factor relative to this decomposition.

The results above can be reformulated as follows:

Proposition 6.2. *Given two differential star products $*$ and $*'$, the zeroth order term of Deligne's relative class $t(*', *) = \sum_{r\ge 0} \nu^r t^r(*', *)$ is given by*

$$t^0(*', *) = -2(C'^-_2)^\# + 2(C^-_2)^\#.$$

It follows from what we did before that the association to a differential star product of $(C^-_2)^\#$ and $d(*)$ completely determines its equivalence class. Let us recall that $d^0(*) = [\omega]$ and $d^1(*) = 0$ and let us observe that if C_1 is just half the Poisson bracket, then $C^-_2(u, v) = A(X_u, X_v)$ where A is a closed 2-form and $(C^-_2)^\# = [A]$ so it "is" the skew-symmetric part of C_2.

We want to now define a class $c(*)(\nu)$ which will determine the equivalence class of $*$ and be equivariant with respect to a change of parameter. By this, we mean the following: consider a star product $*$ defined by

$$u * v = u.v + \sum_{r\ge 1} \nu^r C_r(u, v),$$

where $C^-_1(u, v) = \frac{1}{2}\{u, v\}$ and consider its class $c(*)(\nu)$.

Consider a change of parameter $f(\nu) = \nu + \sum_{r\ge 2} \nu^r f_r$, where $f_r \in \mathbb{R}$ and let $*'$ be the star product obtained from $*$ by this change of parameter, i.e.

$$\begin{aligned} u *' v &= u.v + \sum_{r\ge 1} (f(\nu))^r C_r(u, v) \\ &= u.v + \nu C_1(u, v) + \nu^2 (C_2(u, v) + f_2 C_1(u, v)) + \cdots \end{aligned}$$

Equivariance is the requirement that

$$c(*')(\nu) = c(*)(f(\nu)).$$

Remark that if

$$D_\alpha = \nu \frac{\partial}{\partial \nu} + \mathcal{L}_{X_\alpha} + D^1_\alpha(\nu)$$

is a local derivation of $*$, then

$$D'_\alpha = \frac{f(\nu)}{f'(\nu)} \frac{\partial}{\partial \nu} + \mathcal{L}_{X_\alpha} + D^1_\alpha(f(\nu))$$

is a local derivation of $*'$. Hence a local ν-Euler derivation of $*'$ is given by

$$\tilde{D}_\alpha = \nu\frac{\partial}{\partial\nu} + \frac{\nu f'(\nu)}{f(\nu)}[\mathcal{L}_{X_\alpha} + D^1_\alpha(f(\nu))]$$

since $(\nu f'(\nu)/f(\nu)) = 1 + f_2\nu + \cdots$ With this choice, if $(D_\beta - D_\alpha)(\nu) = (1/\nu)\mathrm{ad}_* d_{\beta\alpha}(\nu)$, one has

$$\begin{aligned}(\tilde{D}_\beta - \tilde{D}_\alpha)(\nu) &= \frac{\nu f'(\nu)}{f(\nu)}(D_\beta - D_\alpha)(f(\nu))\\ &= \frac{\nu f'(\nu)}{(f(\nu))^2}\,\mathrm{ad}_{*'} d_{\beta\alpha}(f(\nu)) = \frac{1}{\nu}\,\mathrm{ad}_{*'}\tilde{d}_{\beta\alpha}(\nu),\end{aligned}$$

with $\tilde{d}_{\beta\alpha}(\nu) = \nu^2 f'(\nu)/(f(\nu))^2 d_{\beta\alpha}(f(\nu))$. From this, we get

$$d(*')(\nu) = \frac{\nu^2 f'(\nu)}{(f(\nu))^2}d(*)(f(\nu)). \tag{10}$$

Let us suppose that $c(*)$ is a solution of

$$\frac{\partial}{\partial\nu}c(*)(\nu) = \frac{d(*)}{\nu^2}.$$

This defines $c(*)$ up to its zeroth order term

$$c(*)(\nu) = \frac{-[\omega]}{\nu} + c(*)^0 + \nu d^2(*) + \cdots + \frac{\nu^k}{k}d^{k+1}(*) + \cdots$$

and Eq. (10) becomes: $(\partial/\partial\nu)c(*')(\nu) = (\partial/\partial\nu)(c(*)(f(\nu))$.

Since

$$\begin{aligned}c(*)(f(\nu)) &= \frac{-[\omega]}{f(\nu)} + c(*)^0 + f(\nu)d^2(*) + \cdots\\ &= \frac{-[\omega]}{\nu}(1 - f_2\nu + (f_2^2 - f_3)\nu^2 + \cdots) + c(*)^0 + \nu d^2(*) + \cdots\\ &= \frac{-[\omega]}{\nu} + f_2[\omega] + c(*)^0 + \nu(d^2(*) + (f_3 - f_2^2)[\omega]) + \cdots\end{aligned}$$

we shall have equivariance of $c(*)$ under a change of parameter if and only if $c(*')^0 = c(*)^0 + f_2[\omega]$. Since $C'^-_2 = C^-_2 - (f_2/2)[\omega]$ (indeed, $\{u, v\} = -\omega(X_u, X_v)$), this is achieved for $c(*)^0 = -2(C^-_2)^\#$.

Definition 6.3. The *characteristic class* $c(*)$ of a differential star product $*$ on (M, ω) is the element of the affine space $(-[\omega]/\nu) + H^2(M;\mathbb{R})[\![\nu]\!]$ defined by

$$c(*)^0 = -2(C^-_2)^\#, \qquad \frac{\partial}{\partial\nu}c(*)(\nu) = \frac{1}{\nu^2}d(*).$$

Theorem 6.4. *The characteristic class has the following properties:*
- *The relative class is given in terms of the characteristic class by*

$$t(*', *) = c(*') - c(*). \tag{11}$$

- *The map C from equivalence classes of star products on (M, ω) to the affine space $-([\omega]/\nu) + H^2(M; \mathbb{R})[\![\nu]\!]$ mapping $[*]$ to $c(*)$ is a bijection – provided one knows that there exists a differential star product on (M, ω).*
- *If $\psi : M \to M'$ is a diffeomorphism and if $*$ is a star product on (M, ω) then $u *' v = (\psi^{-1})^*(\psi^* u * \psi^* v)$ defines a star product $*' = (\psi^{-1})^* *$ on (M', ω'), where $\omega' = (\psi^{-1})^*\omega$. The characteristic class is natural relative to diffeomorphisms*

$$c((\psi^{-1})^* *) = (\psi^{-1})^* c(*). \tag{12}$$

- *Consider a change of parameter $f(\nu) = \sum_{r \geq 1} \nu^r f_r$, where $f_r \in \mathbb{R}$ and $f_1 \neq 0$ and let $*'$ be the star product obtained from $*$ by this change of parameter, i.e. $u *' v = u.v + \sum_{r \geq 1} (f(\nu))^r C_r(u, v) = u.v + f_1 \nu C_1(u, v) + \nu^2((f_1)^2 C_2(u, v) + f_2 C_1(u, v)) + \cdots$. Then $*'$ is a differential star product on (M, ω') where $\omega' = (1/f_1)\omega$ and we have equivariance under a change of parameter*

$$c(*')(\nu) = c(*)(f(\nu)). \tag{13}$$

Remark 6.5. It is shown in [6] that $c(*)$ is the characteristic class introduced by Fedosov as the de Rham class of the curvature of a generalised connection (up to a sign and factors of 2 coming from the assumption that the skew-symmetric part of C_1 is taken here to be half the Poisson bracket). The fact that $d(*)$ and $(C_2^-)^\#$ completely characterise the equivalence class of a star product is also proven by Čech methods in [7].

7. The existence of deformations

The method of De Wilde and Lecomte [9] for proving the existence of a star product on any symplectic manifold employs the same techniques that we have been using in the previous sections. For completeness we include a proof here as refined by De Wilde in [7].

Theorem 7.1. *Given a class $c \in H^2(M; \mathbb{R})[\![\nu]\!]$ there exists a star product $*$ with $c(*) = c$.*

Proof. Given a characteristic class $c = \sum_{r \geq 0} \nu^r c_r$ we recursively build a star product $*$ with C_1 given by half the Poisson bracket and $C_2^{-\#} = -\frac{1}{2} c_0$ such that its intrinsic derivation-related Deligne class is $d = \nu^2(\partial/\partial\nu)c$. The method consists in building, at the same time, a family of local ν-Euler derivations D_α of this star product on the open set U_α

$$D_\alpha = \nu \frac{\partial}{\partial \nu} + X_\alpha + D'_\alpha.$$

where X_α is a chosen conformal vector field on U_α ($\mathcal{L}_{X_\alpha}\omega = \omega$), and D'_α is a formal differential operator vanishing on constants of the form $\sum_{r\geq 1} \nu^r D_{\alpha r}$. We have assumed – to be in the correct equivalence class – that

$$C_1(u,v) = \tfrac{1}{2}\{u,v\}, \qquad C_2(u,v) - C_2(v,u) = A(X_u, X_v),$$

where A is a closed 2-form in a given de Rham class (minus the 0-term in the characteristic class) and that, on $U_\alpha \cap U_\beta$, we have

$$D_\beta - D_\alpha = \frac{1}{\nu}\,\mathrm{ad}_*\, d_{\beta\alpha},$$

where $d_{\beta\alpha} \in N_{\alpha\beta}[\![\nu]\!]$ are such that on $U_\alpha \cap U_\beta \cap U_\gamma$ $d_{\alpha\gamma} + d_{\gamma\beta} + d_{\beta\alpha} = d_{\gamma\beta\alpha} \in \mathbb{R}[\![\nu]\!]$ is defining a 2-cocycle whose Čech class $[d_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\nu]\!]$ is the class d.

The construction is done inductively.

Suppose we have a star product at order K, i.e. $u*v = \sum_{r\leq K} \nu^r C_r(u,v)$ with $(u*v)*w = u*(v*w)$ at order K, with $C_0(u,v) = uv$, $C_1(u,v) = \frac{1}{2}\{u,v\}$, and the skew-symmetric part of C_2 given as above in terms of A. Suppose also that we have a family of local derivations on U_α at order $K-1$ given by

$$D_\alpha = \nu\frac{\partial}{\partial\nu} + X_\alpha + \sum_{1\leq r\leq K-1} \nu^r D_{\alpha r},$$

such that, at order $K-1$ on $U_\alpha \cap U_\beta$:

$$D_\beta - D_\alpha = \frac{1}{\nu}\,\mathrm{ad}_*\, d_{\beta\alpha}^{(K-2)}, \tag{14}$$

where $d_{\beta\alpha}^{(K-2)}$ is the truncation at order $K-2$ of $d_{\beta\alpha}$, i.e.

$$d_{\beta\alpha}^{(K-2)} = \sum_{0\leq r\leq K-2} \nu^r d_{\beta\alpha}^r.$$

Note that, for this induction, we can assume $K \geq 3$. Indeed, choose a symplectic connection ∇ (∇ is torsion-free and $\nabla\omega = 0$) and define

$$u*v = uv + \frac{1}{2}\nu\{u,v\} + \nu^2\left(\frac{1}{8}P^2(u,v) + \frac{1}{2}A(X_u,X_v)\right),$$

where P^2 is the covariant square of the Poisson bracket given by

$$P^2(u,v) = \Lambda^{i_1 j_1}\Lambda^{i_2 j_2}\nabla^2_{i_1 i_2} u \nabla^2_{j_1 j_2} v.$$

Then $*$ is a star product at order 2. It can always be extended to order 3 (see below; the skew part of E_3 vanishes since P^2 is symmetric and A is closed). On the other hand $\mathcal{L}_{X_\alpha} P^2 = \mathcal{L}_{X_\beta} P^2$ on $U_\alpha \cap U_\beta$ since $X_\alpha - X_\beta$ is symplectic and one can find (again, see below; the corresponding A_2 is symmetric) a differential operator R such that $\partial R = \mathcal{L}_{X_\alpha} P^2 + 2P^2$. $D_\alpha = \nu(\mathrm{d}/\mathrm{d}\nu) + \mathcal{L}_{X_\alpha} + \nu D^1_\alpha + \nu^2 R$ is a derivation at order 2 (where $D^1_\alpha(u) = A(X_\alpha, X_u)$) which satisfies

$$
\begin{aligned}
(D_\alpha - D_\beta)(u) &= \mathcal{L}_{X_\alpha} u + \nu D^1_\alpha(u) - \mathcal{L}_{X_\beta} u - \nu D^1_\beta(u) \\
&= \{d^0_{\alpha\beta}, u\} + \nu A(X_{d^0_{\alpha\beta}}, X_u) \\
&= \nu^{-1} \operatorname{ad}_* d^0_{\alpha\beta}(u)
\end{aligned}
$$

at order 2.

Define, in this setting,

$$E(u, v, w) = (u * v) * w - u * (v * w) \tag{15}$$

and write $E(u, v, w) = \sum_r \nu^r E_r(u, v, w) \forall u, v, w \in N$. The fact that we have a star product at order K means that $E_r = 0$, $\forall r \le K$. Define also

$$A_\alpha(u, v) = D_\alpha u * v + u * D_\alpha v - D_\alpha(u * v) \tag{16}$$

and write similarly $A_\alpha(u, v) = \sum_r \nu^r A_{\alpha r} \forall u, v \in N_\alpha$. The fact that the D_α are local derivations at order $K - 1$ means that $A_{\alpha r} = 0, \forall r \le K - 1$.

We have

$$
\begin{aligned}
u * E(v, w, x) - E(u * v, w, x) + E(u, v * w, x)& \\
- E(u, v, w * x) + E(u, v, w) * x = 0&
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
&u * A_\alpha(v, w) - A_\alpha(u * v, w) + A_\alpha(u, v * w) - A_\alpha(u, v) * w \\
&\quad D_\alpha E(u, v, w) - E(D_\alpha u, v, w) - E(u, D_\alpha v, w) - E(u, v, D_\alpha w).
\end{aligned}
\tag{18}
$$

Relation (17) at order $K + 1$ implies that E_{K+1} is a Hochschild 3-cocycle for N (with the associative structure given by the usual product of functions) so

$$E_{K+1}(u, v, w) = B_{K+1}(X_u, X_v, X_w) + \partial C_{K+1}(u, v, w),$$

where B_{K+1} is a 3-form on M and where ∂ denotes the Hochschild coboundary operator on N. The total skew-symmetrisation of relation (17) at order $K + 2$ yields

$$\mathrm{d}B_{K+1} = 0. \tag{19}$$

Relation (18) at order K gives $\partial A_{\alpha K} = 0$ so

$$A_{\alpha K}(u, v) = G_{\alpha K}(X_u, X_v) + \partial D_{\alpha K}(u, v),$$

where $G_{\alpha K}$ is a 2-form on U_α, and the skew-symmetrisation at order $K + 1$ yields

$$\mathrm{d}G_{\alpha K} = 3((K + 1 - 3)B_{K+1} + \mathcal{L}_{X_\alpha} B_{K+1}) \tag{20}$$

using $\mathcal{L}_{X_\alpha} X_u = X_{\mathcal{L}_{X_\alpha} u} - X_u$. Relation (14) implies that

$$\nu(A_\beta(u, v) - A_\alpha(u, v)) = E(d_{\beta\alpha}, u, v) - E(u, d_{\beta\alpha}, v) + E(u, v, d_{\beta\alpha}),$$

so its skew-symmetric part at order $K+1$ gives

$$(G_{\beta K} - G_{\alpha K})(X_u, X_v) = 3B_{K-1}(X_{d^0_{\beta\alpha}}, X_u, X_v),$$

which can be reformulated as

$$G_{\beta K} - G_{\alpha K} = i(X_\beta - X_\alpha)3B_{K-1}. \tag{21}$$

This last formula (21) shows that there is a well-defined 2-form on M

$$G_K = G_{\alpha K} - i(X_\alpha)3B_{K+1}.$$

The relation (20) can be reformulated as

$$\mathrm{d}G_K = 3(K-2)B_{K-1}. \tag{22}$$

Hence, B_{K-1} is an exact 2-form; modifying accordingly the cochain C_K to

$$C'_K(u, v) = C_K(u, v) + \frac{1}{K-2}G_K(X_u, X_v) + F_K(X_u, X_v),$$

where F_K is any closed 2-form on M, we still have a star product at order K but now the corresponding E'_{K+1} is given by $E'_{K+1}(u, v, w) = E_{K+1}(u, v, w) + \frac{1}{2}(\{(C'_K - C_K)(u, v), w\} + (C'_K - C_K)(\{u, v\}, w) - \{u, (C'_K - C_K)(v, w)\} - (C'_K - C_K)(u, \{v, w\}))$ so that the vanishing of its skew-symmetric part gives

$$B'_{K-1} = B_{K+1} - \frac{1}{3(K-2)}\mathrm{d}G_K - \frac{1}{3}\mathrm{d}F_K = 0.$$

Hence E'_{K-1} is a Hochschild coboundary and there exists a C'_{K+1} so that

$$u *' v = \sum_{r \le K-1} \nu^r C_r(u, v) + \nu^K C'_K(u, v) + \nu^{K+1} C'_{K+1}(u, v)$$

is a star product at order $K+1$. Modifying $D_{\alpha K-1}$ by a 1-differential cochain we get new local derivations at order $K-1$ on U_α:

$$D'_\alpha(u) = D_\alpha(u) + \nu^{K-1}R_\alpha(X_u).$$

Now the corresponding $A'_{\alpha K}$ is given by

$$A'_{\alpha K}(u, v) = A_{\alpha K}(u, v) - (K-2)\left(\frac{1}{(K-2)}G_K + F_K\right)(X_u, X_v)$$
$$-\mathcal{L}_{X_\alpha}\left(\frac{1}{(K-2)}G_K + F_K\right)(X_u, X_v) + \frac{1}{2}\mathrm{d}R_\alpha(X_u, X_v),$$

so that, choosing

$$\frac{1}{2}R_\alpha = \mathrm{i}(X_\alpha)F_K + \frac{1}{K-2}\mathrm{i}(X_\alpha)G_K + (K-2)f_{\alpha K},$$

where $f_{\alpha K}$ are 1-forms on U_α chosen in such a way that $\mathrm{d}f_{\alpha K} = F_{K|U_\alpha}$, the skew-symmetric part of $A'_{\alpha K}$ vanishes

$$
\begin{aligned}
G'_{\alpha K} &= G_{\alpha K} - (K-2)\left(\frac{1}{(K-2)}G_K + F_K\right) \\
&\quad - \mathcal{L}_{X_\alpha}\left(\frac{1}{(K-2)}G_K + F_K\right) + \mathrm{d}R_\alpha \\
&= -(K-2)F_K - \mathcal{L}_{X_\alpha}F_K + \frac{1}{2}\mathrm{d}R_\alpha - \frac{1}{K-2}\mathrm{d}i(X_\alpha)G_K \\
&= 0.
\end{aligned}
$$

Hence $A'_{\alpha K}$ is a Hochschild coboundary and there exists a $D'_{\alpha K}$ so that

$$
D'_\alpha = \nu\frac{\partial}{\partial\nu} + X_\alpha + \sum_{1\le r\le K-2} \nu^r D_{\alpha r} + \nu^{K-1}D'_{\alpha K-1} + \nu^K D'_{\alpha K}
$$

is a family of local derivations at order K of our new star product.

Notice furthermore that at order $K-1$ on $U_\alpha \cap U_\beta$ we have

$$
\begin{aligned}
(D'_\beta - D'_\alpha)(u) &= \frac{1}{\nu}\,\mathrm{ad}_{*'}\, d_{\beta\alpha}^{\{K-2\}}(u) - \nu^{K-1}(C'_K - C_K)(d^0_{\beta\alpha}, u) \\
&\quad + \nu^{K-1}(C'_K - C_K)(u, d^0_{\beta\alpha}) \\
&\quad + \nu^{K-1}(D'_{\beta K-1} - D_{\beta K-1} - D'_{\alpha K-1} + D_{\alpha K-1})(u) \\
&= \frac{1}{\nu}\,\mathrm{ad}_{*'}\, d_{\beta\alpha}^{\{K-2\}}(u) + \nu^{K-1}(R_\beta - R_\alpha)(X_u)) \\
&\quad - 2\nu^{K-1}\left(\frac{1}{(K-2)}G_K(X_{d^0_{\beta\alpha}}, X_u) + F_K(X_{d^0_{\beta\alpha}}, X_u)\right) \\
&= \frac{1}{\nu}\,\mathrm{ad}_{*'}\, d_{\beta\alpha}^{\{K-2\}}(u) + \nu^{K-1}2(K-2)(f_{\beta K} - f_{\alpha K})(X_u) \\
&= \frac{1}{\nu}\,\mathrm{ad}_{*'}\, d_{\beta\alpha}^{\{K-1\}}(u)
\end{aligned}
$$

if we choose

$$
f_{\beta K} - f_{\alpha K} = -\frac{1}{2(K-2)}\,\mathrm{d}d^{K-1}_{\beta\alpha}.
$$

i.e. the Čech representation of F_K is $-\frac{1}{2(K-2)}d^{K-1}_{\gamma\beta\alpha}$. Hence, at order K, we have

$$
D'_\beta - D'_\alpha = \frac{1}{\nu}\,\mathrm{ad}_{*'}\, d_{\beta\alpha}^{\{K-1\}} + \nu^K S_{\beta\alpha},
$$

where $S_{\beta\alpha}$ is a 1-differential 1-cochain on $N_{\alpha\beta}$. Remark that $S_{\beta\alpha} + S_{\alpha\gamma} + S_{\gamma\beta} = 0$ hence we can define the 1-differential 1-cochain S_α on N_α by $S_\alpha(u) = \sum_\gamma \theta_\gamma S_{\alpha\gamma}$ so that $S_{\beta\alpha} = S_\beta - S_\alpha$. Modifying D'_α by $D'_\alpha - \nu^K S_\alpha$, we still have a derivation of $*'$ up to order K which now satisfies, at order K;

$$
D'_\beta - D'_\alpha = \frac{1}{\nu}\,\mathrm{ad}_{*'}\, d_{\beta\alpha}^{\{K-1\}}.
$$

Hence the induction can proceed. □

8. Hochschild cohomology of a star-deformed algebra

A differential star product on M defines on $C^\infty(M)[\![\nu]\!]$ the structure of an associative algebra $A = (C^\infty(M)[\![\nu]\!], *)$. As such it can be considered either as an $\mathbb{R}$- or as an $\mathbb{R}[\![\nu]\!]$-algebra. In this section, we study the first and second differential Hochschild cohomology of A viewed as an $\mathbb{R}[\![\nu]\!]$-algebra. The consideration of A as an $\mathbb{R}$-algebra is more complicated and will be looked at in Section 9. p-Cochains C for A are linear in ν and hence determined by p-multilinear maps from N to A, so by a series of p-cochains for N:

$$C(u_1, \dots, u_p) = \sum_{r\geq 0} \nu^r C_r(u_1, \dots, u_p).$$

Definition 8.1. We say a p-cochain for the $\mathbb{R}[\![\nu]\!]$-algebra $A = (N[\![\nu]\!], *)$ is *differential* if each of the C_r is a differential p-cochain on N. We denote the Hochschild coboundary for A by ∂_* and the corresponding cohomology groups computed from differential cochains by $H_\nu^p(A, A)$.

In this setting Proposition 3.5 can be reformulated as follows.

Proposition 8.2. *$H_\nu^1(A, A)$ is isomorphic to $Z^1(M; \mathbb{R}) \oplus \nu H^1(M; \mathbb{R})[\![\nu]\!]$ where $Z^1(M; \mathbb{R})$ denotes the space of closed 1-forms on M.*

To obtain a similar result for the second cohomology group we refine the relationship between the characteristic class and the equivalence class of a star product at a given order in ν to a relationship between a representing 2-form and the cochains of the star product.

Lemma 8.3. *Given a differential star product $*$ on (M, ω), a closed 2-form F on M and an integer $k \geq 0$, one can build a star product $*'$ with $c(*') = c(*) + \nu^k[F]$, such that $*'$ coincides with $*$ up to order $k+1$ and such that their difference at order $k+2$ is $-\frac{1}{2}F(X_u, X_v)$. This $*'$ is unique up to an equivalence of the form $T = I + \nu^{k-2}\mathcal{L}_X + \cdots$, where X is a vector field on M.*

Proof. We fix, as before, a locally finite contractible open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ and choose $F = \mathrm{d}a_\alpha$ on U_α, $a_\beta - a_\alpha = \mathrm{d}a_{\beta\alpha}$ on $U_\alpha \cap U_\beta$ and $a_{\gamma\beta\alpha} = a_{\alpha\gamma} + a_{\gamma\beta} + a_{\beta\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ so that $\{a_{\gamma\beta\alpha}\}$ is a representative of the Čech class corresponding to $[F]$.

A corollary of the proof of Proposition 4.5 is that we can find differential operators T_α on U_α with

$$T_\alpha(u) = u + \nu^{k+1} a_\alpha(X_u) + \cdots$$

so that $T_\beta^{-1} \circ T_\alpha = \exp \mathrm{ad}_* t_{\beta\alpha}$, where the $t_{\beta\alpha} \in N_{\alpha\beta}[\![\nu]\!]$ are of the form

$$t_{\beta\alpha} = \nu^k a_{\beta\alpha} + \cdots$$

and satisfy

$$a_{\gamma\beta\alpha} = t_{\alpha\gamma} \circ_* t_{\gamma\beta} \circ_* t_{\beta\alpha}.$$

The differential star product $*''$ defined on U_α by

$$u *'' v = T_\alpha(T_\alpha^{-1}u * T_\alpha^{-1}v)$$

coincides with $*$ at order $k+1$ and the skew-symmetric part of their difference at order $k+2$ is given by $-\frac{1}{2}F(X_u, X_v)$. Combining with an equivalence $T = I + \nu^{k+2}E$ if the symmetric part of their difference at order $k+2$ is ∂E, we get a $*'$ as stated in the lemma.

Now two such star products are equivalent since they have the same characteristic class. Let $T = \sum_{r\geq j}\nu^r T_r$ be an equivalence between them. If $j \leq k+1$, the equivalence relation at order j shows that T_j is a vector field and the antisymmetric part of the terms of degree $j+1$ show that it is a Hamiltonian vector field $T_j(u)|_{U_\alpha} = \{h_\alpha, u\}$ for some locally defined function h_α. Then $\nu^{-1}\,\mathrm{ad}_*\,h_\alpha$ is globally defined and $(\exp(-\nu^{j-1}\,\mathrm{ad}_*\,h_\alpha)) \circ T = \mathrm{Id} + \mathrm{O}(\nu^{j+1})$ is again an equivalence between our two star products. By induction on j, we can assume that the equivalence is of the form $T = I + \nu^{k+2}\mathcal{L}_X + \cdots$, where X is a vector field on M. □

Given a differential star product $*$ on (M, ω) and a formal series of closed 2-forms on M $F = \nu^k F_k + \sum_{r>k}\nu^r F_r$ (k integer ≥ 0), one can build as above a family $*^s$ of differential star products, depending smoothly on s, such that $c(*^s) = c(*) + s[F]$, such that $*^s$ coincides with $*$ at order $k+1$ in ν for all s and such that their difference at order $k+2$ in ν is $-1/2sF_k(X_u, X_v)$. Define

$$D_F(u, v) = \nu^{-2}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} u *^s v.$$

This is a 2-cocycle for the ν-linear Hochschild cohomology of A. Its lowest order term is $-\frac{1}{2}\nu^k F(X_u, X_v)$. The class of this Hochschild 2-cocycle does not depend on the choice of the smooth family of star products $*^s$ since any other choice corresponds to

$$u\tilde{*}^s v = T_s(T_s^{-1}u *^s T_s^{-1}v),$$

with $T_s = I + \nu^{k-2}\mathcal{L}_{X_s} + \cdots$, and

$$D'(u, v) = \nu^{-2}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 u\tilde{*}^s v = D(u, v) + \partial_* E(u, v),$$

where

$$E(u) = \nu^{-2}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 T_s^{-1}u.$$

This yields a map $D : Z^2(M; \mathbb{R})[\![\nu]\!] \to H_\nu^2(A, A)$.

Let D be a ν-linear differential 2-cocycle for the Hochschild cohomology of A. It is determined by its values on $N \times N$. Assume its lowest order term is $\nu^k D_k$. Looking at $\partial_* D = 0$ at order k and its skew-symmetric part at order $k+1$ shows that there is a closed 2-form F on M such that $D_k(u, v) = F(X_u, X_v) + S(u, v)$ where S is a symmetric cocycle for the Hochschild cohomology of N. Hence S is a coboundary in that cohomology, $S = \partial R$, and $\nu^k\partial_* R$ has lowest order term $\nu^k S$. So D is cohomologous to $D_{-2F} + D'$, with D_F as before for any $F' = \nu^k F + \cdots$ and D' with lowest order term of degree $k+1$. If the closed 2-form is exact, $F = dB$, and if $k > 0$, then $D - \partial_* E$ where $E(u) = \nu^{k-1} B(X_u)$ has lowest order term of degree k which is symmetric, hence can be removed by adding another coboundary. If D is a coboundary and has lowest order term $\nu^k(F(X_u, X_v) + S(u, v))$, then either $k = 0$ and $F = 0$, or $k \geq 1$ and F is exact. This yields the following:

Proposition 8.4. *$H^2_\nu(A, A)$ is canonically isomorphic to*

$$Z^2(M; \mathbb{R}) \oplus \nu H^2(M; \mathbb{R})[\![\nu]\!],$$

where $Z^2(M; \mathbb{R})$ denotes the space of closed 2-forms on M. The isomorphism associates to the class of $F = \sum_{r \geq 0} \nu^r F_r$ where $F_r \in Z^2(M; \mathbb{R})$ the class of the 2-cocycle $D_F(u, v) = \nu^{-2}(\mathrm{d}/\mathrm{d}s)|_0 u *^s v$ where $*^s$ is any family of differential star products, depending smoothly in s, such that $c(*^s) = c(*) + s[F]$, such that $*^s$ coincide with $*$ at order 1 in ν for all s and such that their difference at order 2 in ν is $-\frac{1}{2} s F_0(X_u, X_v)$.

Remark 8.5. If we replace $\mathbb{R}[\![\nu]\!]$ by the ring of formal Laurent polynomials $\mathbb{R}[\nu^{-1}, \nu]\!]$ then we can also subtract an exact term from the leading closed 2-form and we obtain the result of Weinstein and Xu [25] that the ν-linear Hochschild cohomology $H^2_\nu(N[\nu^{-1}, \nu]\!], N[\nu^{-1}, \nu]\!])$ is $H^2(M; \mathbb{R})[\nu^{-1}, \nu]\!]$.

Remark 8.6. Proposition 8.4 is still true if $*^s$ is any family of differential star products depending smoothly on s such that $c(*'_0) = c(*)$ and $(d/\mathrm{d}s)|_0 c(*^s) = [F]$, such that at order 1 $*^s$ coincides with $*$ for all s and at order 2 their difference is $-\frac{1}{2} s F(X_u, X_v)$. In particular, if $(u *^s v)(\nu) = (u * v)(f_s(\nu))$ with $f_0(\nu) = \nu$ and $(\mathrm{d}/\mathrm{d}s)|_0 f_s(\nu) = \nu^2$ then $(\mathrm{d}/\mathrm{d}s)|_0 u *^s v$ corresponds to $\omega + \sum_k \nu^k d_k$ if $d(*) = [\omega] + \sum_k \nu^k d_k$.

9. Derivations and automorphisms of a star product

In this section, we consider a star-deformed algebra as a real algebra and we study its derivations and its automorphisms. We assume throughout this section that the manifold M is connected. The results presented here were obtained with one of our students, Rauch [23].

Definition 9.1. A *derivation* of a differential star product $*$ on (M, ω) is an $\mathbb{R}$-linear map $D : N[\![\nu]\!] \to N[\![\nu]\!]$, continuous in the ν-adic topology (i.e. $D(\sum_r \nu^r u_r)$ is the limit of $\sum_{r \leq N} D(\nu^r u_r)$), such that

$$D(u * v) = Du * v + u * Dv.$$

Note that

$$D(\nu) * u = D(\nu * u) - \nu * D(u) = D(u * \nu) - D(u) * \nu = u * D(\nu)$$

so that $D(\nu)$ must be central and thus $D(\nu) \in \mathbb{R}[\![\nu]\!]$. Hence D restricted to $\mathbb{R}[\![\nu]\!]$ is a derivation $D(g(\nu) = \tilde{f}(\nu)(\partial/\partial\nu) g(\nu)$ where $\tilde{f}(\nu) \in \mathbb{R}[\![\nu]\!]$. Hence $D(g(\nu)u) =$

$\tilde{f}(\nu)(\partial/\partial\nu)g(\nu)u + g(\nu)D(u)$. The term of order zero in ν of the derivation relation implies that $\tilde{f}(\nu) = \nu f(\nu)$. Combining this with some previous results we get the following:

Proposition 9.2. *Any local derivation of a differential star product (on a contractible open set U_α) is of the form*

$$D_\alpha = f(\nu)D'_\alpha + D''_\alpha,$$

where D''_α is a ν-linear local derivation of $*$, i.e.

$$D''_\alpha = \frac{1}{\nu}\,\mathrm{ad}_* d_\alpha, \quad d_\alpha \in N_\alpha[\![\nu]\!],$$

where $f(\nu) \in \mathbb{R}[\![\nu]\!]$ and where D'_α is a chosen local ν-Euler derivation.

Definition 9.3. An *isomorphism* A from a differential star product $*$ on (M,ω) to a differential star product $*'$ on (M',ω') is an $\mathbb{R}$-linear bijective map $A : N[\![\nu]\!] \to N'[\![\nu]\!]$, continuous in the ν-adic topology, such that

$$A(u * v) = Au *' Av.$$

Notice that if A is such an isomorphism, then $A(\nu)$ is central for $*'$ so that $A(\nu) = f(\nu)$, where $f(\nu) \in \mathbb{R}[\![\nu]\!]$ is without constant term to get the ν-adic continuity. Let us denote by $*''$ the differential star product on $(M, \omega_1 = (1/f_1)\omega)$ obtained by a change of parameter

$$u *'' v_\nu = u * v_{f(\nu)} = F(F^{-1}u * F^{-1}v)$$

for $F : N[\![\nu]\!] \to N[\![\nu]\!] : \sum_r \nu^r u_r \mapsto \sum_r f(\nu)^r u_r$. Define $A' : N[\![\nu]\!] \to N'[\![\nu]\!]$ by $A = A' \circ F$. Then A' is a ν-linear isomorphism between $*''$ and $*'$:

$$A'(u *'' v) = A'u *' A'v.$$

At order zero in ν this yields

$$A'_0(u.v) = A'_0u.A'_0v$$

so that there exists a diffeomorphism $\psi : M' \to M$ with $A'_0u = \psi^* u$. The skew-symmetric part of the isomorphism relation at order 1 in ν implies that $\psi^*\omega_1 = \omega'$. Let us denote by $*'''$ the differential star product on (M, ω_1) obtained by pullback via ψ of $*'$:

$$u *''' v = (\psi^{-1})^*(\psi^* u *' \psi^* v)$$

and define $B : N[\![\nu]\!] \to N[\![\nu]\!]$ so that $A' = \psi^* \circ B$. Then B is ν-linear, starts with the identity and

$$B(u *'' v) = Bu *''' Bv$$

so that B is an equivalence – in the usual sense – between $*''$ and $*'''$. Hence, we summarise:

Proposition 9.4. *Any isomorphism between two differential star products is the combination of a change of parameter and a ν-linear isomorphism. Any ν-linear isomorphism* *between two star products $*$ on (M, ω) and $*'$ on (M', ω') is the combination of the action on functions of a symplectomorphism $\psi : M' \to M$ and an equivalence between $*$ and the pullback via ψ of $*'$. In particular, it exists if and only if those two star products are equivalent, i.e. if and only if $(\psi^{-1})^* c(*') = c(*)$, where here $(\psi^{-1})^*$ denotes the action on the second de Rham cohomology space.*

This implies immediately:

Corollary 9.5. *Two star products $*$ on (M, ω) and $*'$ on (M', ω') are isomorphic if and only if there exist $f(\nu) = \sum_{r \geq 1} \nu^r f_r \in \mathbb{R}[\![\nu]\!]$ with $f_1 \neq 0$ and $\psi : M' \to M$, a symplectomorphism, such that $(\psi^{-1})^* c(*')(f(\nu)) = c(*)(\nu)$. In particular* [13]*: if $H^2(M; \mathbb{R}) = \mathbb{R}[\omega]$ then there is only one star product up to equivalence and change of parameter.*

Remark 9.6. See also Omori et al. [21] who show that when reparametrisations are allowed then there is only one star product on $\mathbb{C}P^n$.

In particular, Proposition 9.4 gives:

Corollary 9.7. *A symplectomorphism ψ of a symplectic manifold (M, ω) can be extended to a ν-linear automorphism of a given differential star product on (M, ω) if and only if $(\psi)^* c(*) = c(*)$.*

Notice that this is always the case if ψ can be connected to the identity by a path of symplectomorphisms ([11].

10. On Deligne's definition of a deformation

In this section we fill in some of the steps to get from the definition of a deformation in Deligne's paper [6] to the usual one considered in the first part of these notes.

10.1. Deformations of $C^\infty(M)$

We shall work just with real smooth functions on a manifold, the other cases considered by Deligne can easily be handled in a similar manner. Denote by N the $\mathbb{R}$-algebra of smooth functions on a manifold and consider a pair (A, φ) consisting of an $\mathbb{R}[\![\nu]\!]$-algebra A and a surjective $\mathbb{R}$-algebra homomorphism

$$\varphi : A \to N$$

such that Ker $\varphi = \nu A$.

N is commutative so $\varphi(a)\varphi(b) = \varphi(b)\varphi(a)$ and hence $ab - ba \in \nu A$. Thus there is an element which we denote by $\nu^{-1}(ab - ba)$ in A which is unique up to an element of A annihilated by φ. We shall shortly assume that A is free over $\mathbb{R}[\![\nu]\!]$ so we feel free to abuse notation for now.

Proposition 10.1. *There is a unique Poisson structure on N such that*

$$\varphi(\nu^{-1}(ab - ba)) = \{\varphi(a), \varphi(b)\}$$

Proof. If $u, v \in N$ pick $a, b \in A$ such that $\varphi(a) = u$, $\varphi(b) = v$. Then define $\{u, v\} = \varphi(\nu^{-1}(ab-ba))$. This is well defined since if $\varphi(a') = u$ and $\varphi(b') = v$ then and $a - a' = \nu c$ and $b - b' = \nu d$ so

$$\begin{aligned}(ab - ba) &= (a' + \nu c)(b' + \nu d) - (b' + \nu d)(a' + \nu c)\\ &= a'b' - b'a' + \nu(cb' - b'c + a'd - da') + \nu^2(cd - dc).\end{aligned}$$

Thus

$$\begin{aligned}\varphi(\nu^{-1}(ab - ba)) &= \varphi(\nu^{-1}(a'b' - b'a') + cb' - b'c + a'd - da' + \nu(cd - dc))\\ &= \varphi(\nu^{-1}(a'b' - b'a')).\end{aligned}$$

The bracket $\{\,,\,\}$ on N obviously satisfies the Bianchi identity. It is a Poisson bracket since if $\varphi(a) = u$, $\varphi(b) = v$, $\varphi(c) = w$ then $\varphi(ab) = uv$ so

$$\begin{aligned}\{uv, w\} &= \varphi(\nu^{-1}(abc - cab))\\ &= \varphi(\nu^{-1}(abc - acb + acb - cab))\\ &= \varphi(a\nu^{-1}(bc - cb) + \nu^{-1}(ac - ca)b)\\ &= \varphi(a)\varphi(\nu^{-1}(bc - cb)) + \varphi(\nu^{-1}(ac - ca))\varphi(b)\\ &= u\{v, w\} + \{u, w\}v. \qquad \square\end{aligned}$$

Definition 10.2. We say A is *ν-adically complete* if, given any sequence a_r, $r \geq 0$ of elements of A, there is an element $a \in A$ such that for each $k \geq 0$ there is an element $b_{k+1} \in A$ with

$$a - \sum_{r=0}^{k} \nu^r a_r = \nu^{k-1} b_{k+1}.$$

where a is denoted by $\sum_{r\geq 0} \nu^r a_r$.

Thus A is closed under taking formal power series in its elements. An example is, of course, $N[\![\nu]\!]$.

In order that the algebra A looks like $N[\![\nu]\!]$ we assume that A is ν-adically complete and has an $\mathbb{R}$-linear subspace mapped bijectively onto N by φ which, together with ν, freely generates A in the ν-adic topology. In other words, we have an $\mathbb{R}$-linear map $\rho : N \to A$ with $\varphi \circ \rho = \mathrm{Id}$ and such that the map $\widehat{\rho} : N[\![\nu]\!] \to A$ given by

$$\widehat{\rho}\left(\sum_{r\geq 0} \nu^r u_r\right) = \sum_{r\geq 0} \nu^r \rho(u_r)$$

induces a bijection of $N[\![\nu]\!]$ onto A ($\widehat{\rho}$ exists from the ν-adic completeness of A). In this case, if $u, v \in N$ then there are functions $C_r(u, v) \in N$ with

$$\rho(u)\rho(v) = \sum_{r\geq 0} \nu^r \rho(C_r(u, v)). \tag{23}$$

We call such a map ρ a section of (A, φ).

Proposition 10.3. *Given a section ρ and the C_r as above then*

$$C_0(u, v) = uv, \qquad C_1(u, v) - C_1(v, u) = \{u, v\}.$$

Proof.

$$\begin{aligned} C_0(u, v) &= \varphi(\rho(C_0(u, v)) \\ &= \varphi\left(\sum_{r\geq 0} \nu^r \rho(C_r(u, v))\right) \\ &= \varphi(\rho(u)\rho(v)) = \varphi(\rho(u))\varphi(\rho(v)) = uv \end{aligned}$$

and

$$\begin{aligned} C_1(u, v) - C_1(v, u) &= \varphi(\rho(C_1(u, v)) - \varphi(\rho(C_1(v, u)) \\ &= \varphi\left(\sum_{r\geq 1} \nu^{r-1}\rho(C_r(u, v)) - \sum_{r\geq 1} \nu^{r-1}\rho(C_r(v, u))\right) \\ &= \varphi\left(\nu^{-1}\sum_{r\geq 0} \nu^r \rho(C_r(u, v)) - \nu^{-1}\sum_{r\geq 0} \nu^r \rho(C_r(v, u))\right) \\ &= \varphi(\nu^{-1}\rho(u)\rho(v) - \nu^{-1}\rho(v)\rho(u)) \\ &= \{u, v\}. \qquad \square \end{aligned}$$

If we define $\varphi_0 : N[\![\nu]\!] \to N$ by

$$\varphi_0\left(\sum_{r\geq 0} \nu^r u_r\right) = u_0$$

then $\varphi \circ \widehat{\rho} = \varphi_0$.

Fixing a section ρ we can transfer the algebra structure from A to $N[\![\nu]\!]$ using $\widehat{\rho}$ and denote the resulting multiplication by $*$:

$$u * v = \widehat{\rho}^{-1}\left(\widehat{\rho}(u)\widehat{\rho}(v)\right).$$

If we restrict this to elements u, v of N then

$$u * v = \sum_{r>0} \nu^r C_r(u, v), \qquad u, v \in N.$$

In view of Proposition 10.3, $*$ is a star product on N given by the cochains C_r.

If ρ' is a second section and $*'$ the star product it defines

$$u *' v = \sum_{r \geq 0} \nu^r C'_r(u, v), \quad u, v \in N,$$

where

$$\rho'(u)\rho'(v) = \sum_{r \geq 0} \nu^r \rho'(C'_r(u, v)),$$

then

$$\rho'(u) = \sum_{r \geq 0} \nu^r \rho(T_r(u))$$

for some sequence of functions $T_r(u)$. Applying φ to this equation we get

$$u = T_0(u)$$

so that

$$T_0 = \mathrm{Id}\,.$$

If we set

$$T(u) = u + \sum_{r \geq 1} \nu^r T_r(u),$$

then

$$\begin{aligned}\sum_{r \geq 0} \nu^r \rho(T_r(u)) \sum_{s \geq 0} \nu^s \rho(T_s(v)) &= \sum_{p \geq 0} \nu^p \sum_{r+s=p} \rho(T_r(u))\rho(T_s(v)) \\ &= \sum_{p \geq 0} \nu^p \sum_{r+s=p} \sum_{q \geq 0} \nu^q \rho(C_q(T_r(u), T_s(v)))\end{aligned}$$

so that

$$\begin{aligned}\rho'(u)\rho'(v) &= \sum_{r \geq 0} \nu^r \rho'(C'_r(u, v)) \\ &= \sum_{r \geq 0} \nu^r \sum_{s \geq 0} \nu^s \rho(T_s(C'_r(u, v))).\end{aligned}$$

Comparing coefficients of powers of ν we obtain

$$\begin{aligned}\sum_{r+s=t} T_s(C'_r(u, v)) &= \sum_{p+q=t} \sum_{r+s=p} C_q(T_r(u), T_s(v)) \\ &= \sum_{r+s+q=t} C_q(T_r(u), T_s(v)).\end{aligned}$$

A straightforward calculation now shows that

$$T(u *' v) = T(u) * T(v)$$

so that $*'$ and $*$ are equivalent.

To obtain a differential star product we have to assume that there is a section ρ which makes the C_r into bidifferential operators. We shall call such a ρ a *differential section*. Two different sections ρ_i, $i = 1, 2$ give rise to equivalent star products. Theorem 2.22 says that if the two star products are differential then they are equivalent by a differential operator. Thus if a deformation A gives rise to a differential star product, then all the differential star products it gives rise to are in a single differential equivalence class.

Definition 10.4. A (*formal, differential*) *deformation* of a symplectic manifold (M, ω) is a pair (A, φ) consisting of an $\mathbb{R}[\![\nu]\!]$-algebra A and an $\mathbb{R}$-algebra epimorphism $\varphi : A \to N = C^{\infty}(M)$ such that
1 Ker $\varphi = \nu A$;
2 the Poisson bracket induced on N by Proposition 10.1 coincides with that coming from the symplectic structure;
3 there are $\mathbb{R}$-linear maps $\rho : N \to A$ (called sections of φ) whose image freely generates A as a ν-adically complete $\mathbb{R}[\![\nu]\!]$-algebra;
4 the cochains $C_r(u, v)$ associated to such a section ρ by (23) are bidifferential operators (ρ is a differential section).

In the usual definition of star product 1 is assumed to be an identity for the star multiplication. This can be made to be the case here.

Proposition 10.5. *Let $\varphi : A \to N$ be a deformation then A has a unity 1_A in $\varphi^{-1}(1)$. There exist sections ρ with $\rho(1) = 1_A$.*

Proof. Pick a in A with $\varphi(a) = 1$. Then left (or right) multiplication by a is a bijection of A with itself. This follows easily by an induction after choosing a section ρ to represent elements of A as formal power series. Thus a must be left multiplication of some element 1_A by a: $a = a1_A$. Then any element b is ca for some c so $b1_A = ca1_A = ca = b$. Similarly there is an element $1'_A$ with $a = 1'_A a$. Then $1'_A b = b$ for any b. Thus $1'_A = 1'_A 1_A = 1_A$ so 1_A is a two-sided unity for A. Given any section ρ' of φ then $c = \rho'(1)$ is invertible with $\varphi(c) = \varphi(c^{-1}) = 1$. Thus $\rho(u) = c^{-1}\rho'(u)$ is a section with $\rho(1) = 1_A$. □

Remark 10.6. If we take a section ρ respecting unities then the cochains C_r it defines vanish on constants for $r \geq 1$. The corresponding star product then satisfies $1 * u = u * 1 = u$ for all u in $N[\![\nu]\!]$.

Definition 10.7. Two (A_i, φ_i), $i = 1, 2$ deformations of N are said to be *equivalent* if there is an $\mathbb{R}[\![\nu]\!]$-algebra isomorphism $\psi : A_1 \to A_2$ continuous in the ν-adic topology such that $\varphi_2 \circ \psi = \varphi_1$.

Here continuity means that ψ commutes with taking formal power series

$$\psi\left(\sum_{r\geq 0} \nu^r a_r\right) = \sum_{r\geq 0} \nu^r \psi(a_r).$$

Proposition 10.8. *Equivalent deformations of N induce the same Poisson bracket on N.*

Proof. Let (A_i, φ_i), $i = 1, 2$ be two deformations of N with $\psi : A_1 \to A_2$ an isomorphism such that $\varphi_2 \circ \psi = \varphi_1$, and $\{\,,\,\}_i$ the induced Poisson brackets. If $u, v \in N$, pick $a_i, b_i \in A_i$ with $\varphi_i(a_i) = u$, $\varphi_i(b_i) = v$ then $\psi(a_1) - a_2 = \nu c$, $\psi(b_1) - b_2 = \nu d$. Thus $\psi(a_1 b_1 - b_1 a_1) = a_2 b_2 - b_2 a_2 + \nu e$ for $e = a_2 d - d a_2 + c b_2 - b_2 c + \nu(cd - dc)$. Then $\varphi_2(e) = 0$ so $\{u, v\}_2 = \varphi_2(\nu^{-1}(a_2 b_2 - b_2 a_2)) = \varphi_2(\nu^{-1}\psi(a_1 b_1 - b_1 a_1)) = \varphi_2(\psi(\nu^{-1}(a_1 b_1 - b_1 a_1))) = \varphi_1(\nu^{-1}(a_1 b_1 - b_1 a_1)) = \{u, v\}_1$. □

Given equivalent deformations (A_i, φ_i), $i = 1, 2$ of N with $\psi : A_1 \to A_2$ such that $\varphi_2 \circ \psi = \varphi_1$, if ρ_1 is a section of φ_1 then $\rho_2 = \psi \circ \rho_1$ is a section of φ_2. Let $*^{(i)}$, $i = 1, 2$ be the star products the two sections define with cochains $C_r^{(i)}$. Then

$$\begin{aligned}
\sum_{r\geq 0} \nu^r \rho_2(C_r^{(2)}(u, v)) &= \rho_2(u)\rho_2(v) \\
&= \psi(\rho_1(u)\rho_1(v)) \\
&= \psi\left(\sum_{r\geq 0} \nu^r \rho_1(C_r^{(1)}(u, v))\right) \\
&= \sum_{r\geq 0} \nu^r \rho_2(C_r^{(1)}(u, v))
\end{aligned}$$

so $C_r^{(2)} = C_r^{(1)}$. If we had used a different section of φ_2 we would have obtained equivalent cochains so equivalent deformations lead to equivalent star products. This suggests the following theorem.

Theorem 10.9. *Equivalent deformations (A_i, φ_i), $i = 1, 2$ of N induce equivalent star products on $N[\![\nu]\!]$. This induces a bijection between the set of equivalence classes of deformations and the set of equivalence classes of star products.*

Proof. What remains to be proved is that the map constructed in the previous paragraph is bijective. To see this we take a star product $*$ arising from sections ρ_i of deformations (A_i, φ_i), $i = 1, 2$ so that

$$\sum_{r\geq 0} \nu^r \rho_i(C_r(u, v)) = \rho_i(u)\rho_i(v), \quad i = 1, 2.$$

Define $\psi : A_1 \to A_2$ by $\psi = \widehat{\rho_2} \circ \widehat{\rho_1}^{-1}$. ψ is clearly a bijection from A_1 to A_2 and if we define $\varphi_0\left(\sum_{r\geq 0} \nu^r u_r\right) = u_0$ then $\varphi_i \circ \widehat{\rho_i} = \varphi_0$, $i = 1, 2$ so

$$\varphi_2 \circ \psi = \varphi_2 \circ \widehat{\rho_2} \circ \widehat{\rho_1}^{-1} = \varphi_0 \circ \widehat{\rho_1}^{-1} = \varphi_1.$$

It remains to show that ψ is an algebra isomorphism (it is $\mathbb{R}[\![\nu]\!]$-linear from the way it is defined). But this follows since both algebras A_i are isomorphic to $(N[\![\nu]\!], *)$ and ψ intertwines the two isomorphisms. □

10.2. Deformations of the sheaf C^∞_M

The fact that $A \xrightarrow{\varphi} C^\infty(M)$ has differential sections allows us to localise elements of A on M. More precisely, if ρ and ρ' are two differential sections then $\widehat{\rho}$ and $\widehat{\rho'}$ differ by a formal differential equivalence T such that $\widehat{\rho'} = \widehat{\rho} \circ T$. If $a = \widehat{\rho}\left(\sum_{r\geq 0} \nu^r u_r\right) = \widehat{\rho'}\left(\sum_{r\geq 0} \nu^r u'_r\right)$ then $\sum_{r\geq 0} \nu^r u_r = T\left(\sum_{r\geq 0} \nu^r u'_r\right)$. Thus all functions u_r vanish on an open set U if and only if all the u'_r do. Hence we can unambiguously make the following definition:

Definition 10.10. We say that $a \in A$ vanishes on the open set U if there is a differential section ρ such that $a = \widehat{\rho}\left(\sum_{r\geq 0} \nu^r u_r\right)$ with all $u_r|_U = 0$. We say $a = b$ on U if $a - b$ vanishes on U.

Definition 10.11. Given $x \in M$ we say a and b are *equivalent* at x if there is an open set U containing x with $a = b$ on U. The equivalence class of a at x is called the germ of a at x and denoted by $a[x]$. $\mathcal{A}_x$ denotes the set of germs of elements of A at x.

We claim that $\mathcal{A}_x$ is an algebra over $\mathbb{R}[\![\nu]\!]$ and has an $\mathbb{R}$-algebra homomorphism φ onto C^∞_x, the algebra of germs of smooth functions at x. Here we use the fact that the sheaf of smooth functions is soft (mou): every local germ is the germ of a globally defined smooth function.

Lemma 10.12. *If a vanishes on U and $b \in A$ then ab vanishes on U.*

Proof. Picking a differential section ρ, the multiplication in A is given by a differential star product and if one of the formal power series vanishes on an open set so does the product since it is given by differential cochains. □

This allows us to define the product of two germs as the germ of the product of any two elements in the germs and $\mathcal{A}_x$ becomes an algebra as claimed. To construct a sheaf we topologise the disjoint union $\mathcal{A} = \dot{\cup}_x \mathcal{A}_x$ by taking as a base for a topology the sets of germs of elements a in A over open sets U in M. $\mathcal{A}$ is then a sheaf of algebras and we have a morphism $\varphi : \mathcal{A} \to C^\infty_M$ of sheaves of algebras which is clearly surjective. The kernel of $\varphi : \mathcal{A}_x \to C^\infty_x$ is found as follows: Suppose $\varphi(a[x]) = 0$ then $a = \widehat{\rho}\left(\sum_{r\geq 0} \nu^r u_r\right)$ with $u_0[x] = 0$ If we set $b = a - \rho(u_0)$ then b has the same germ as a at x and $b = \nu\widehat{\rho}\left(\sum_{r\geq 0} \nu^r u_{r+1}\right) = \nu c$. Thus $a[x] = \nu c[x]$. Hence Ker $\varphi = \nu\mathcal{A}$.

If $\mathcal{A}$ is the sheaf associated to a differential deformation $\varphi : A \to C^\infty(M)$, then consider an element a of A. Taking the germ $a[x]$ of a at each $x \in M$, gives a global section $\widehat{a}$ of $\mathcal{A}$.

Lemma 10.13. *The map $a \mapsto \widehat{a}$, just defined, gives a bijection of A with $\Gamma \mathcal{A}$.*

Proof. If $\widehat{a} = 0$ then a is equivalent to 0 on a neighbourhood of each point of M. Pick a differential section ρ, then $a = \widehat{\rho}\left(\sum_{r \geq 0} \nu^r u_r\right)$ with each u_r vanishing on a neighbourhood of each point. It follows that $u_r = 0$, $\forall r \geq 0$ and hence that $a = 0$ so that the map is injective.

For surjectivity, fix a differential section ρ. Then given a section $\widehat{a}$ of $\mathcal{A}$, it determines a sequence of germs of functions u_r at each point and varying the point we get a section of the sheaf of germs of functions. This must come from a global smooth function and hence from some element $\widehat{\rho}\left(\sum_{r \geq 0} \nu^r u_r\right)$ of a. This proves the surjectivity. □

These observations allow us to pass back and forth between the global algebras $\varphi : A \to C^\infty(M)$ to the corresponding sheaves of algebras $\varphi : \mathcal{A} \to C^\infty_M$ (or their presheaves) in such a way that the original algebras are the spaces of global sections.

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